Solving the *Tensor Isomorphism Problem* for special orbits with low rank points

Valerie Gilchrist, Laurane Marco, Christophe Petit, Gang Tang

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Commitment schemes

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Coin flip!

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Does it work over the phone?

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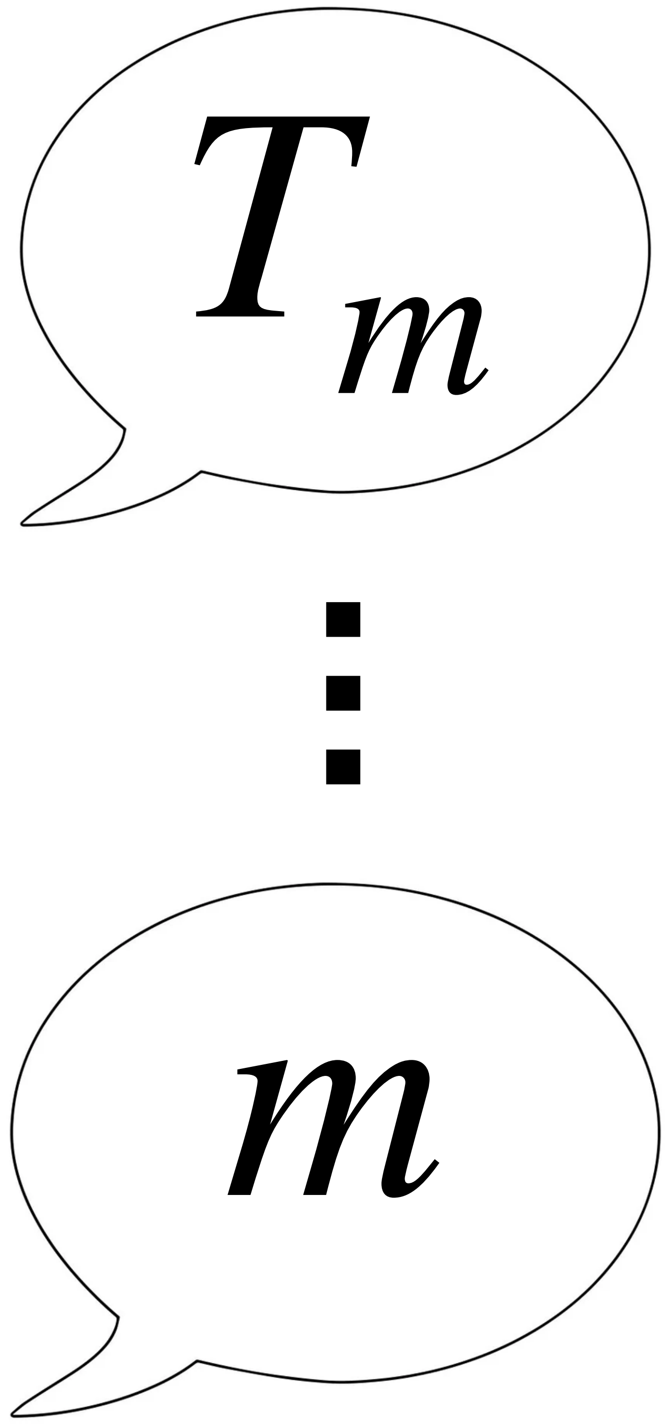


the sender publishes a commitment

*d*epending on m

Later the sender releases m,

and a verifier can check that T_m

was created using m

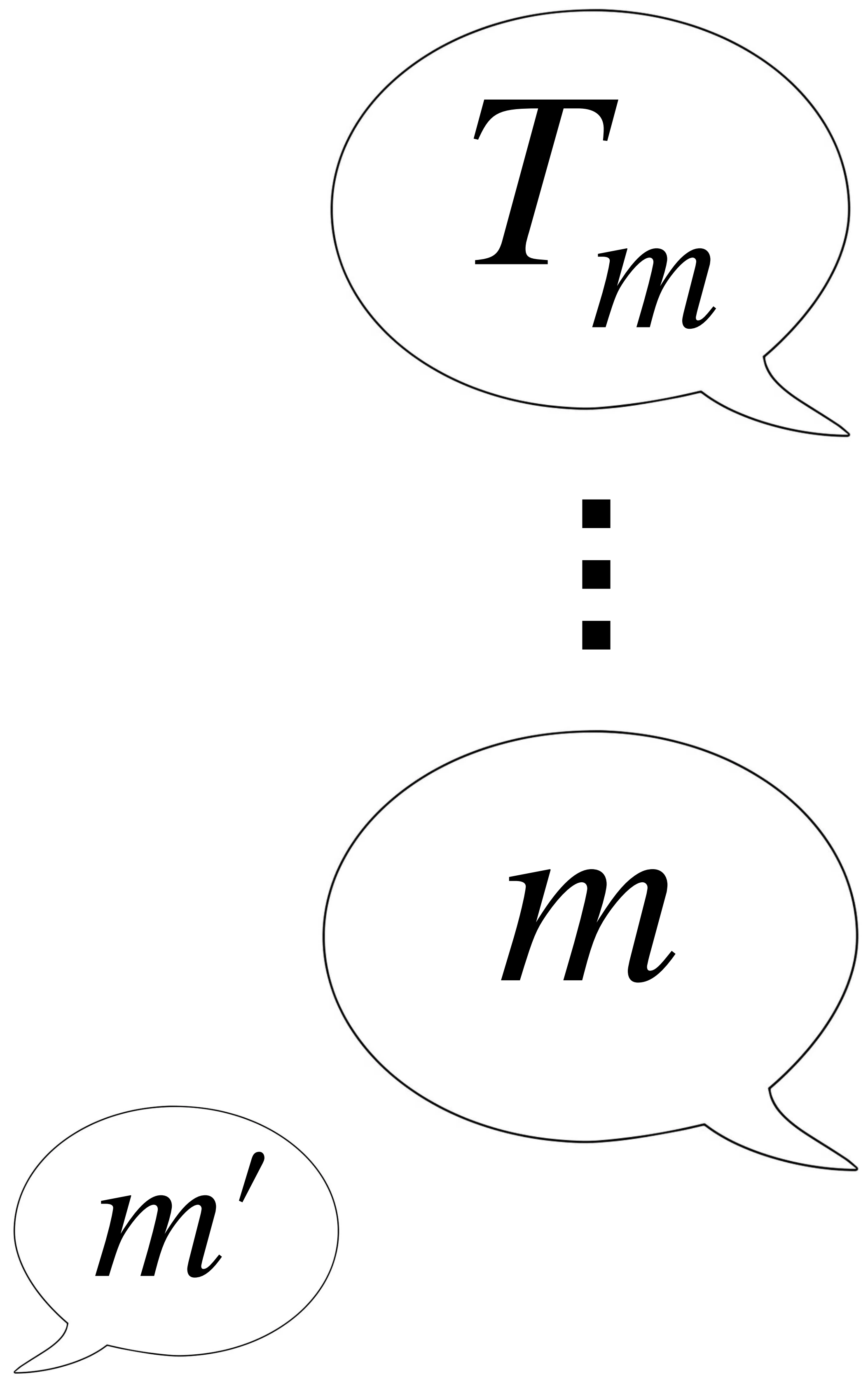
Commitment schemes

The scheme should be ***hiding:***

- the commitment T should leak no information about m

The scheme should be ***binding:***

- no other value $m' \neq m$ should be able to open T



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We would like to compute the tensor $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$ where $\mathbf{u} = [u_1, u_2], \mathbf{v} = [v_1, v_2], \mathbf{w} = [w_1, w_2]$.

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We proceed by first expanding the matrix $\mathbf{v} \cdot \mathbf{w}^T$:

$$\mathbf{v} \cdot \mathbf{w}^T = \begin{bmatrix} v_1 w_1 & v_1 w_2 \\ v_2 w_1 & v_2 w_2 \end{bmatrix}$$

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Now we multiply this matrix by each entry of $\mathbf{u}$, storing them in a list as we go:

$$u_1 \begin{bmatrix} v_1 w_1 & v_1 w_2 \\ v_2 w_1 & v_2 w_2 \end{bmatrix}, u_2 \begin{bmatrix} v_1 w_1 & v_1 w_2 \\ v_2 w_1 & v_2 w_2 \end{bmatrix}$$

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Let t be a tensor. We can always write it as

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➡

$$t = u_1v_1w_1 \cdot e_1 \otimes e_1 \otimes e_1 + u_1v_1w_2 \cdot e_1 \otimes e_1 \otimes e_2 \cdots$$

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$$\begin{aligned} t &= u_1v_1w_1 \cdot e_1 \otimes e_1 \otimes e_1 \\ &\quad + u_1v_1w_2 \cdot e_1 \otimes e_1 \otimes e_2 \cdots \\ &= \sum_{i,j,k} u_iv_jw_k \cdot\ e_i \otimes e_j \otimes e_k \end{aligned}$$

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We can compute the following isomorphism applied to t :

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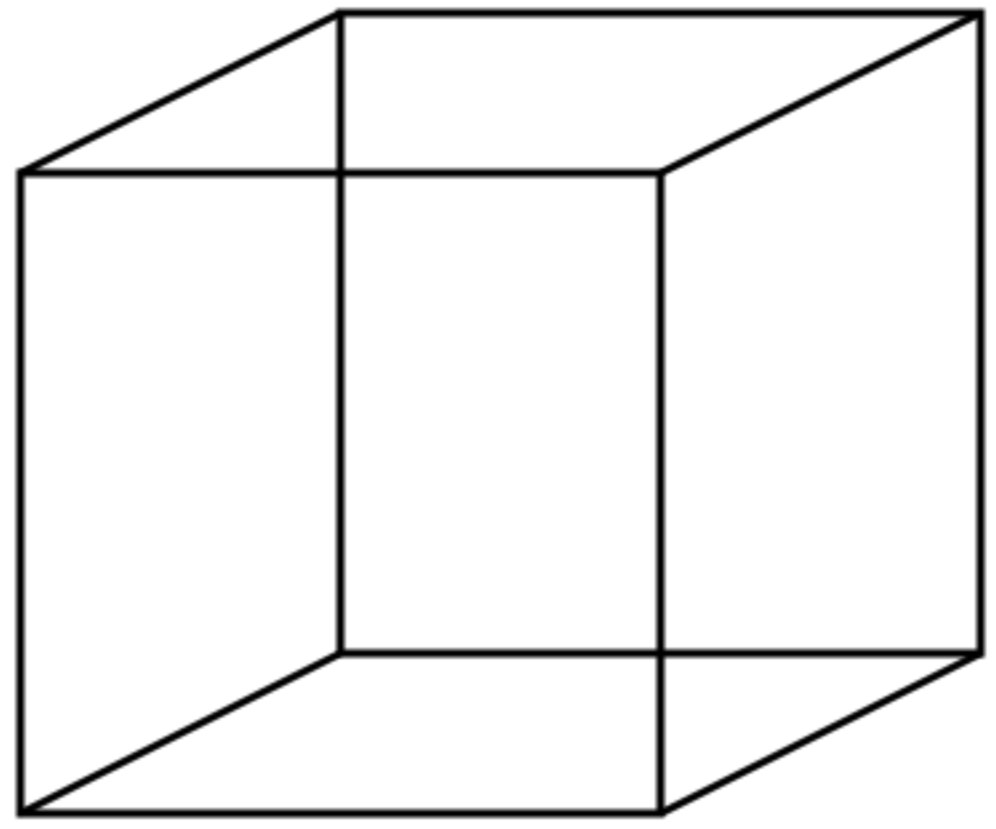

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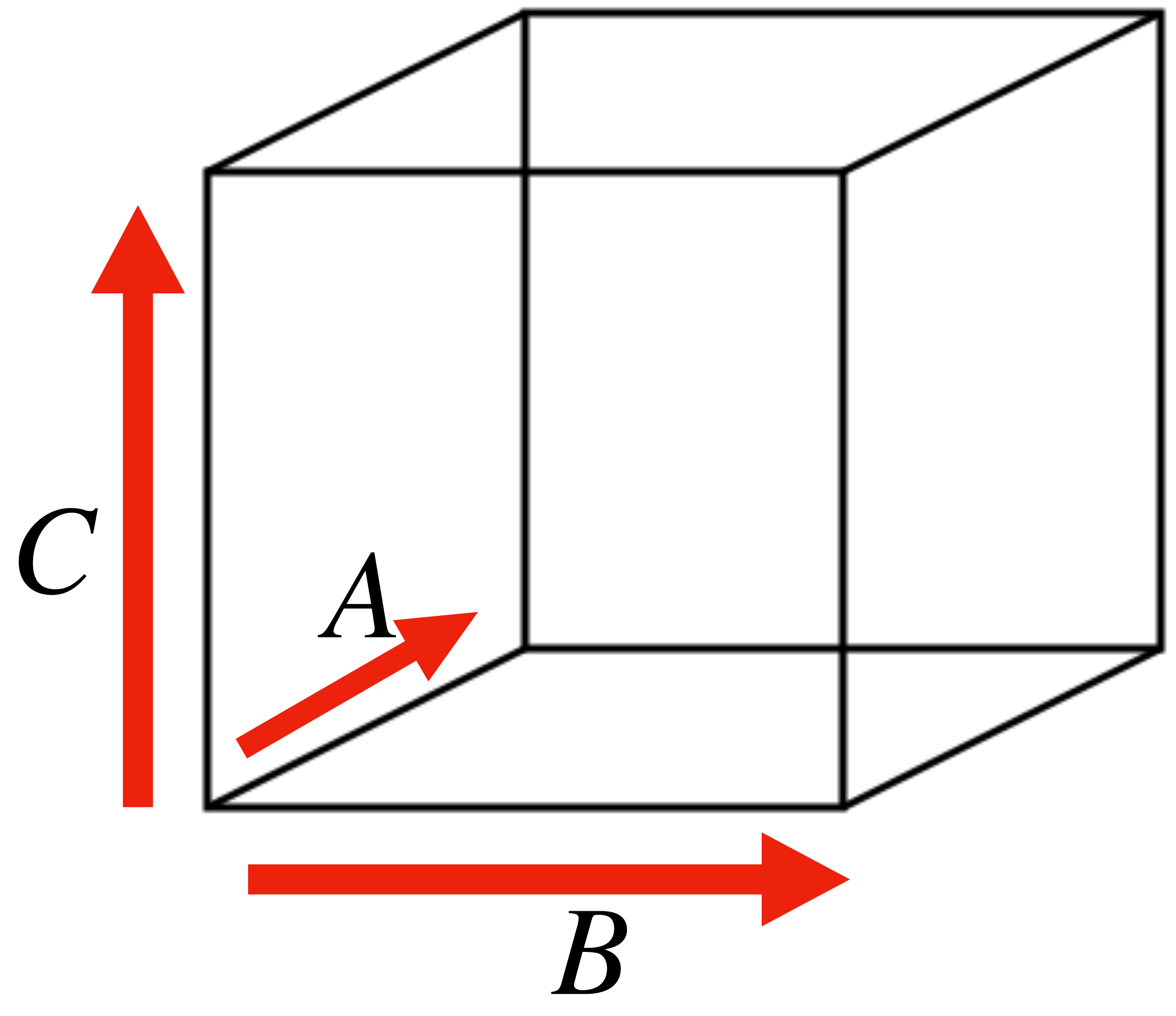

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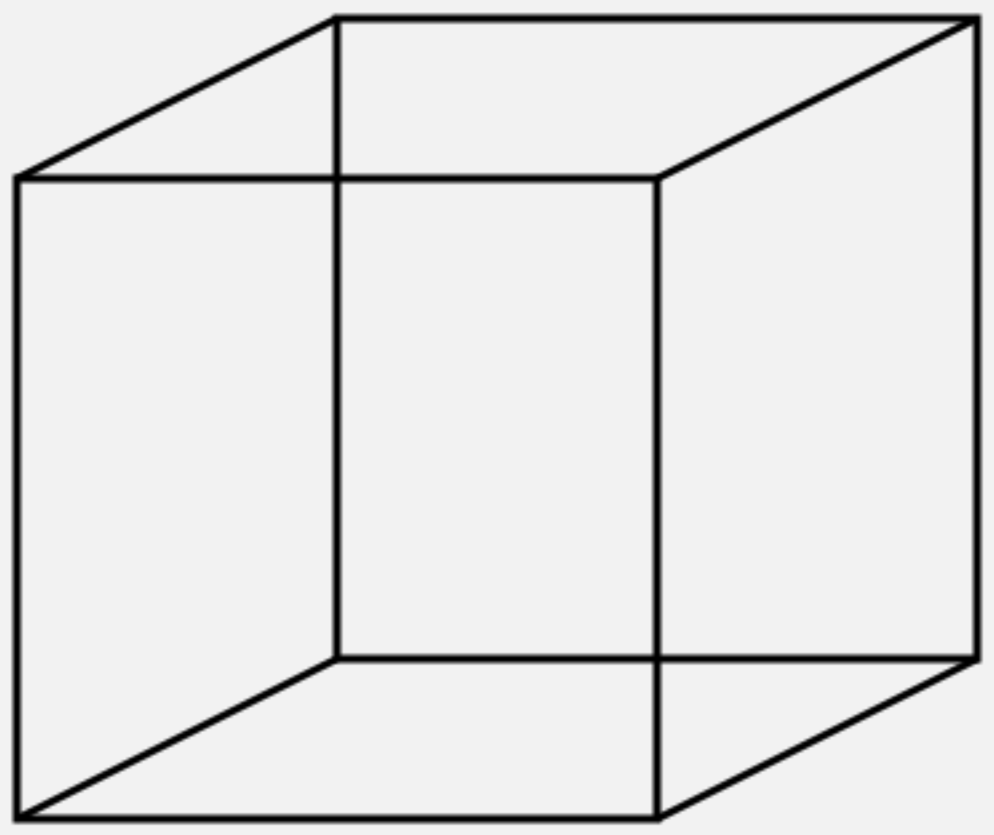

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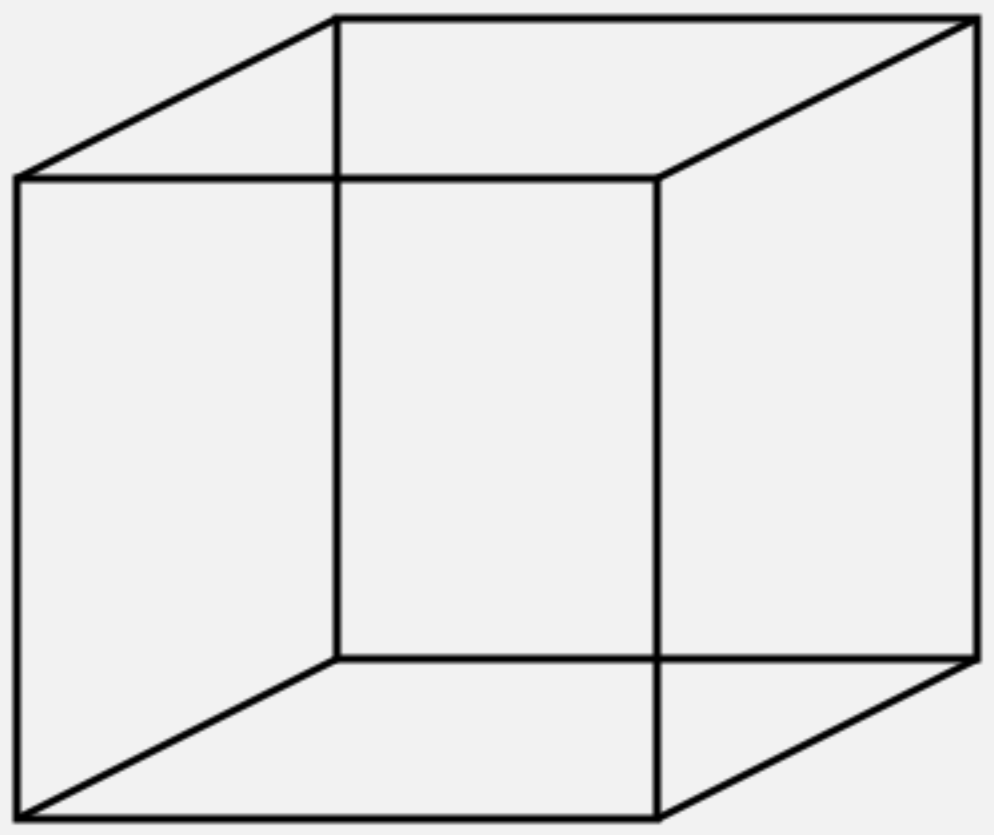

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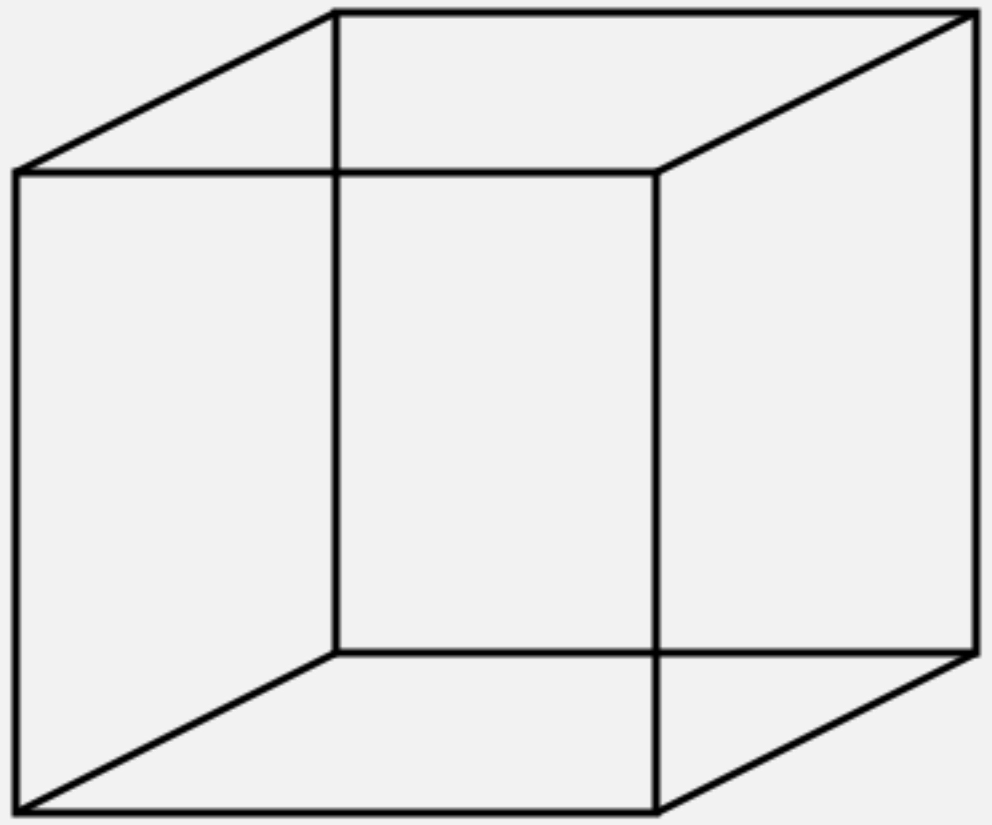

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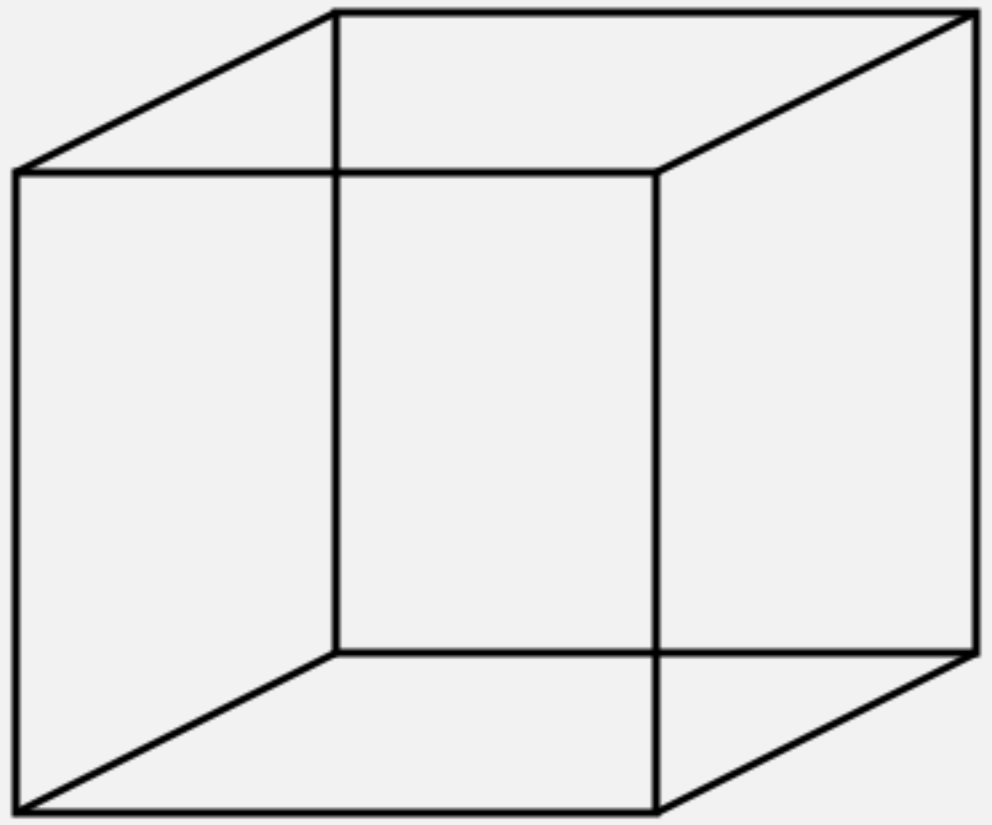

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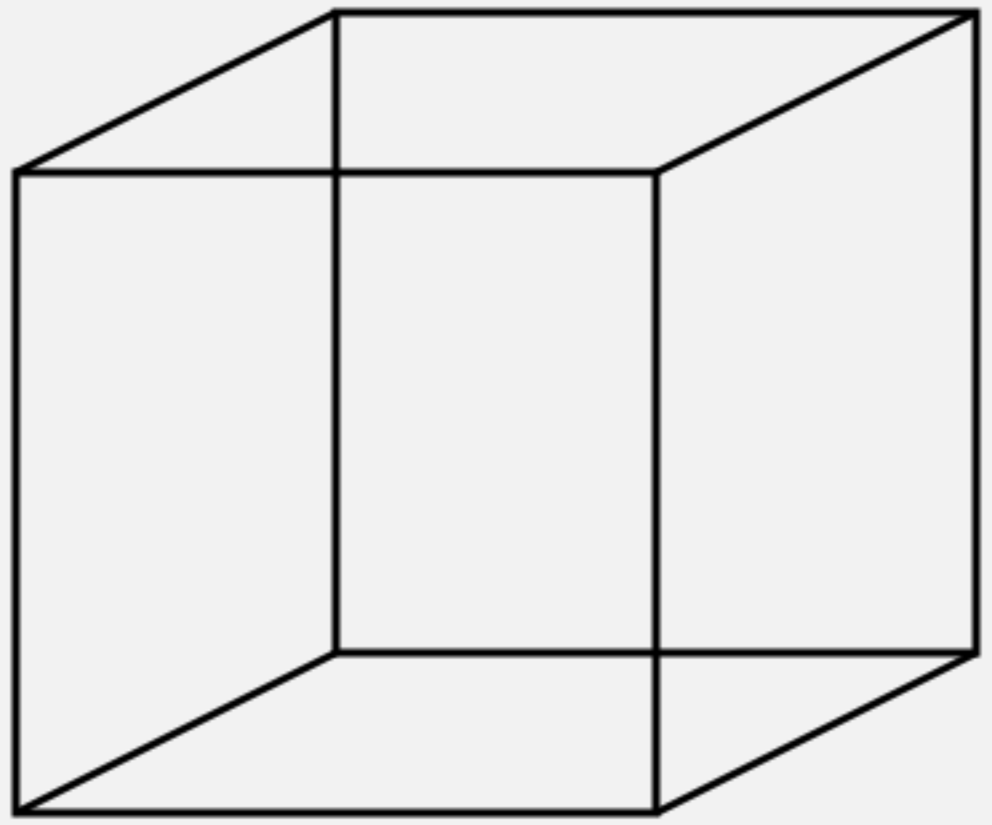

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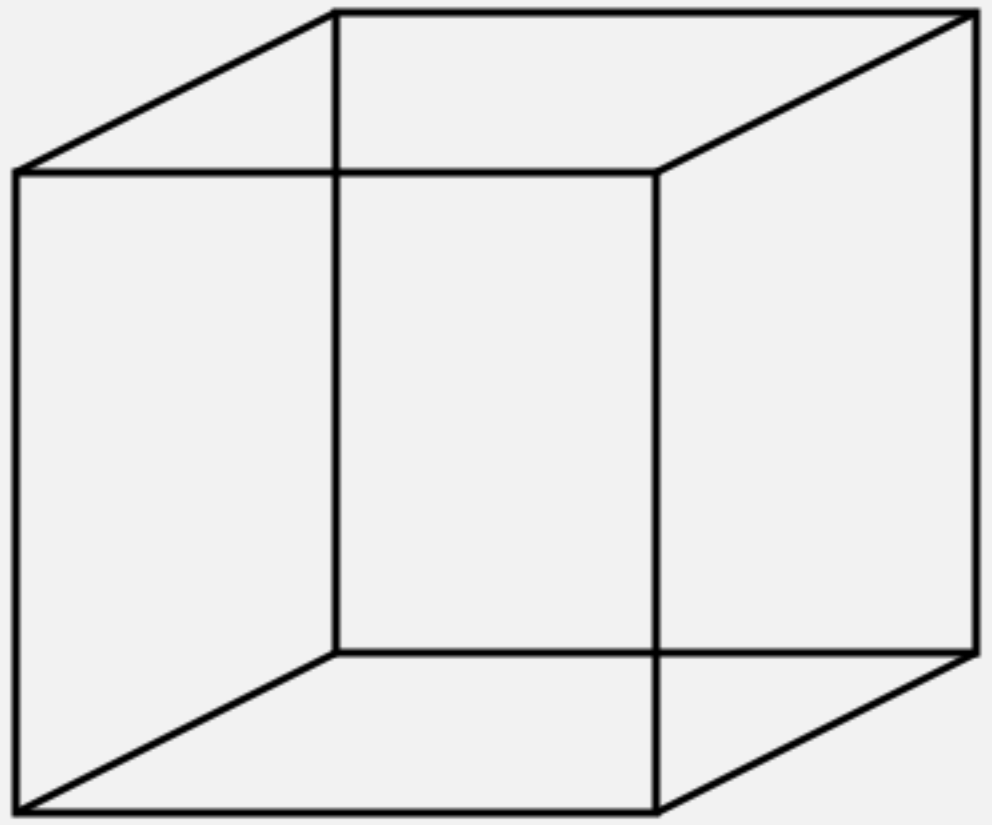

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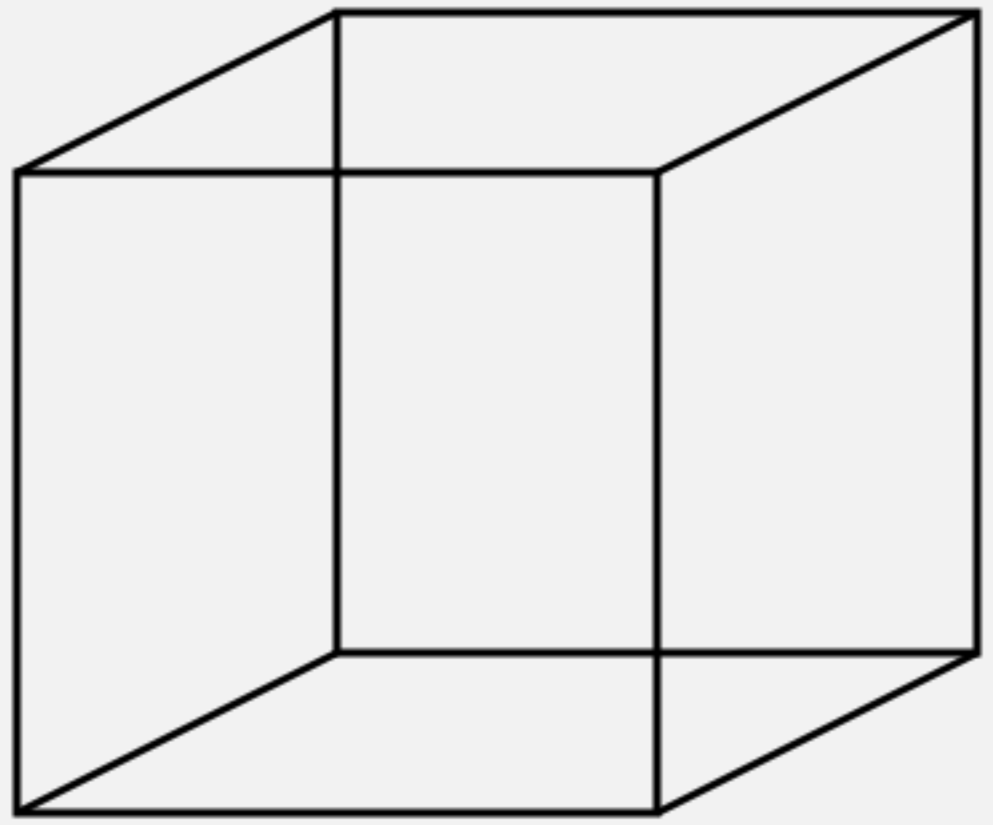

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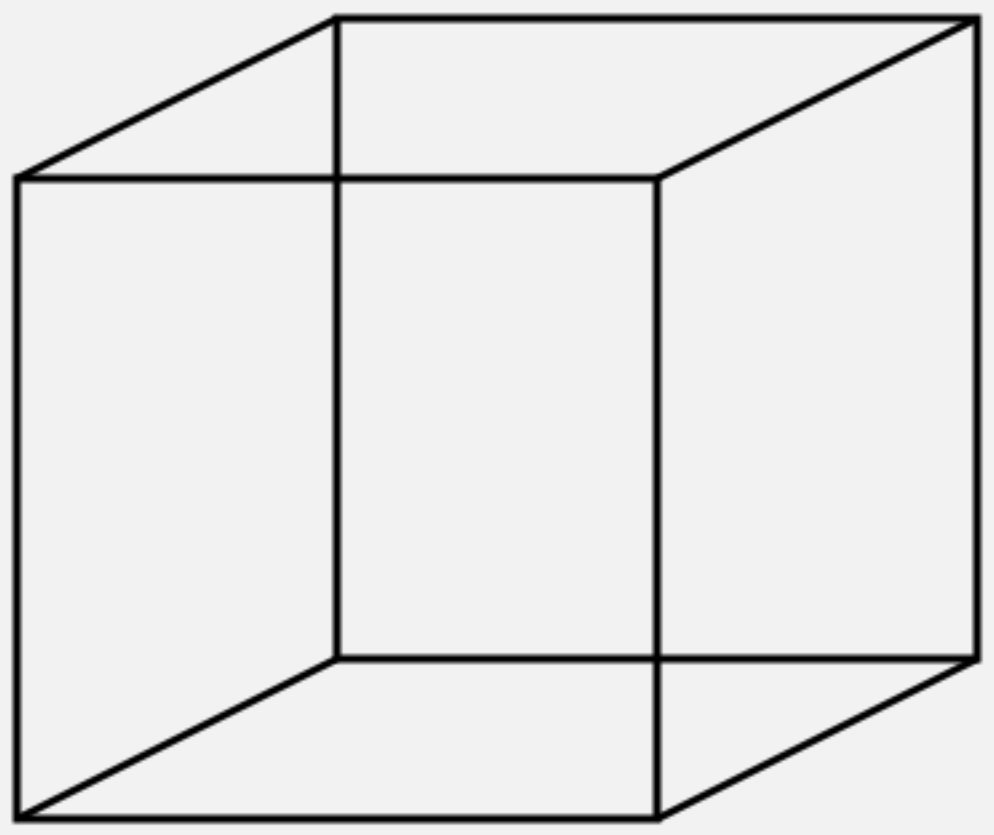

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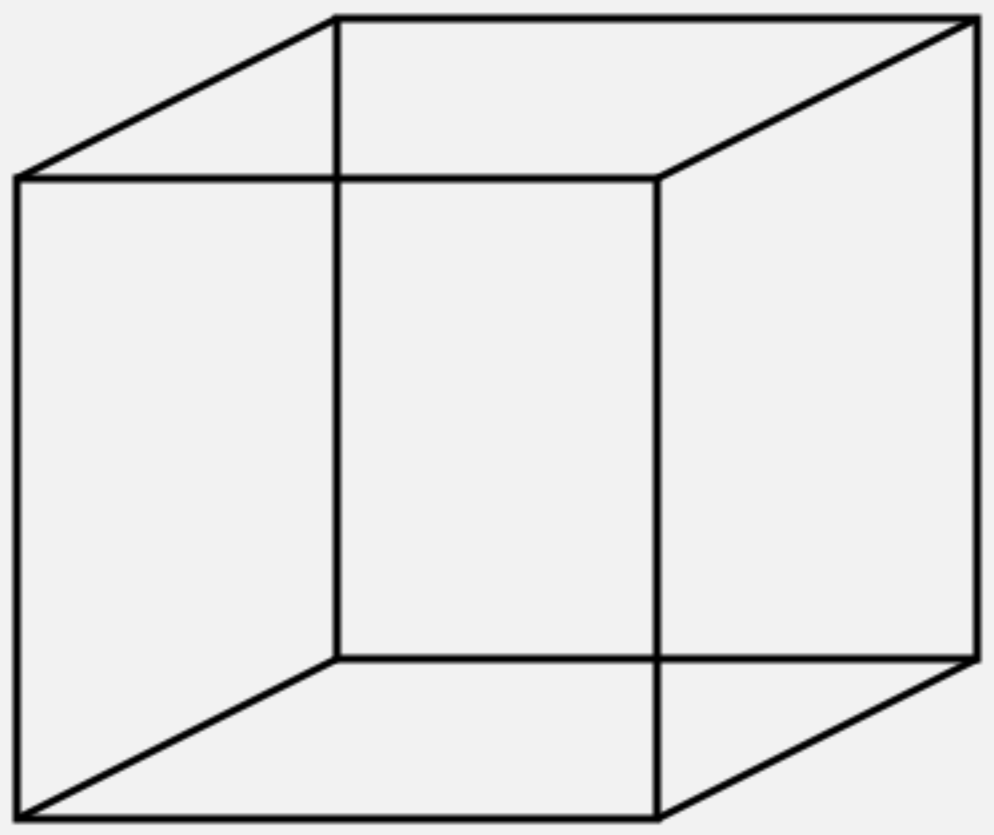

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Decisional Tensor Isomorphism Problem (dTIP):

Given **random** v_0, v_1 **decide** whether there exists

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TI-family

* this is for the case where all the dimensions are n

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Codes are **equivalent** if they generate the same subspace.

Equivalent codes take the form

$$G' = [\lambda_{1,1}G_1 + \cdots \lambda_{1,n}G_n, \cdots \lambda_{n,1}G_1 + \cdots \lambda_{n,n}G_n]$$

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Matrix Code Equivalence (MCE):

Given G, G' compute (if it exists)

(A, B, C) such that $(A, B, C) \star G = G'$

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We say two trilinear forms, φ, ψ, are **equivalent** if there exists some $A \in GL_n(\mathbb{F}_q)$ such that

$$\varphi(u, v, w) = \psi(Au, Av, Aw)$$

TI-family

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Trilinear Form Equivalence (TFE):

Given D, D' compute (if it exists)

(A, B, C) such that $(A, B, C) \star D = D'$

DFG paper

The DFG paper (*Asiacrypt 2023*) seeks to use this hard problem in a commitment scheme

Non-Interactive Commitment from Non-Transitive Group Actions

Giuseppe D'Alconzo[1] [0000−0001−7377−6617], Andrea Flamini[2] [0000−0002−3872−7251], and Andrea Gangemi[2] [0000−0001−9689−8473]

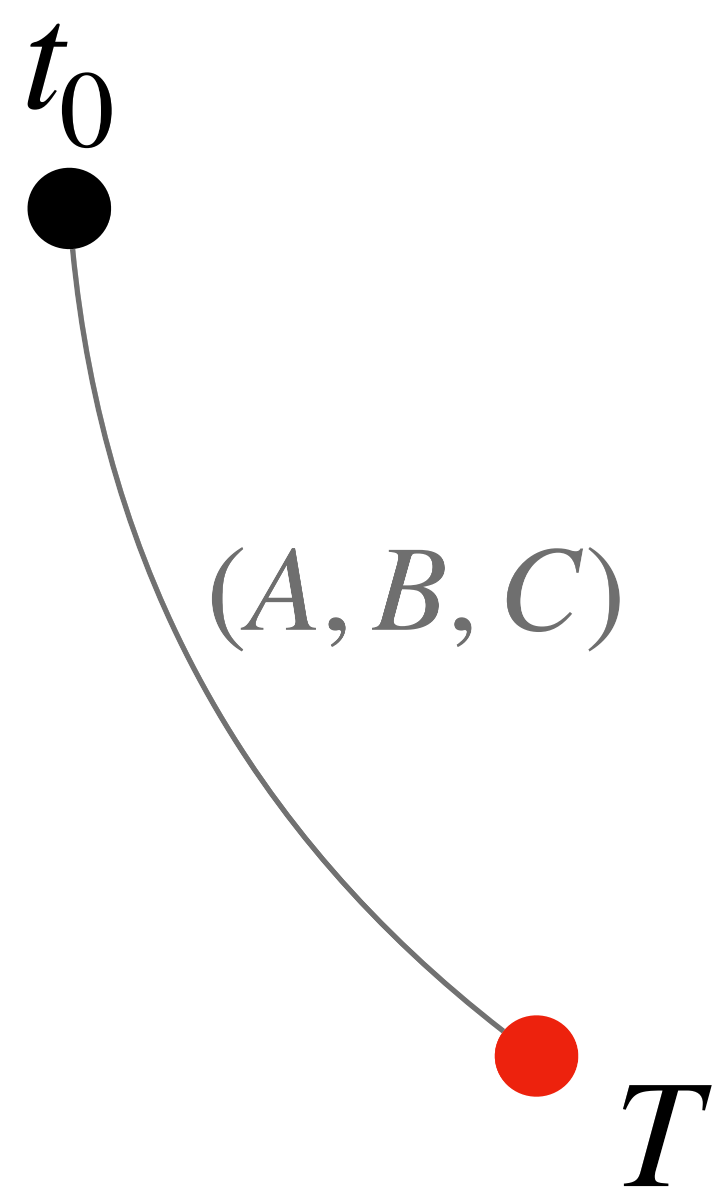
[1] Department of Mathematical Sciences, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
[2] Department of Mathematics, University of Trento, Povo, 38123 Trento, Italy
giuseppe.dalconzo@polito.it, {andrea.flamini,andrea.gangemi}@unitn.it

Abstract. Group actions are becoming a viable option for post-quantum cryptography assumptions. Indeed, in recent years some works have shown how to construct primitives from assumptions based on isogenies of ellip-

DFG paper

DFG paper

DFG paper



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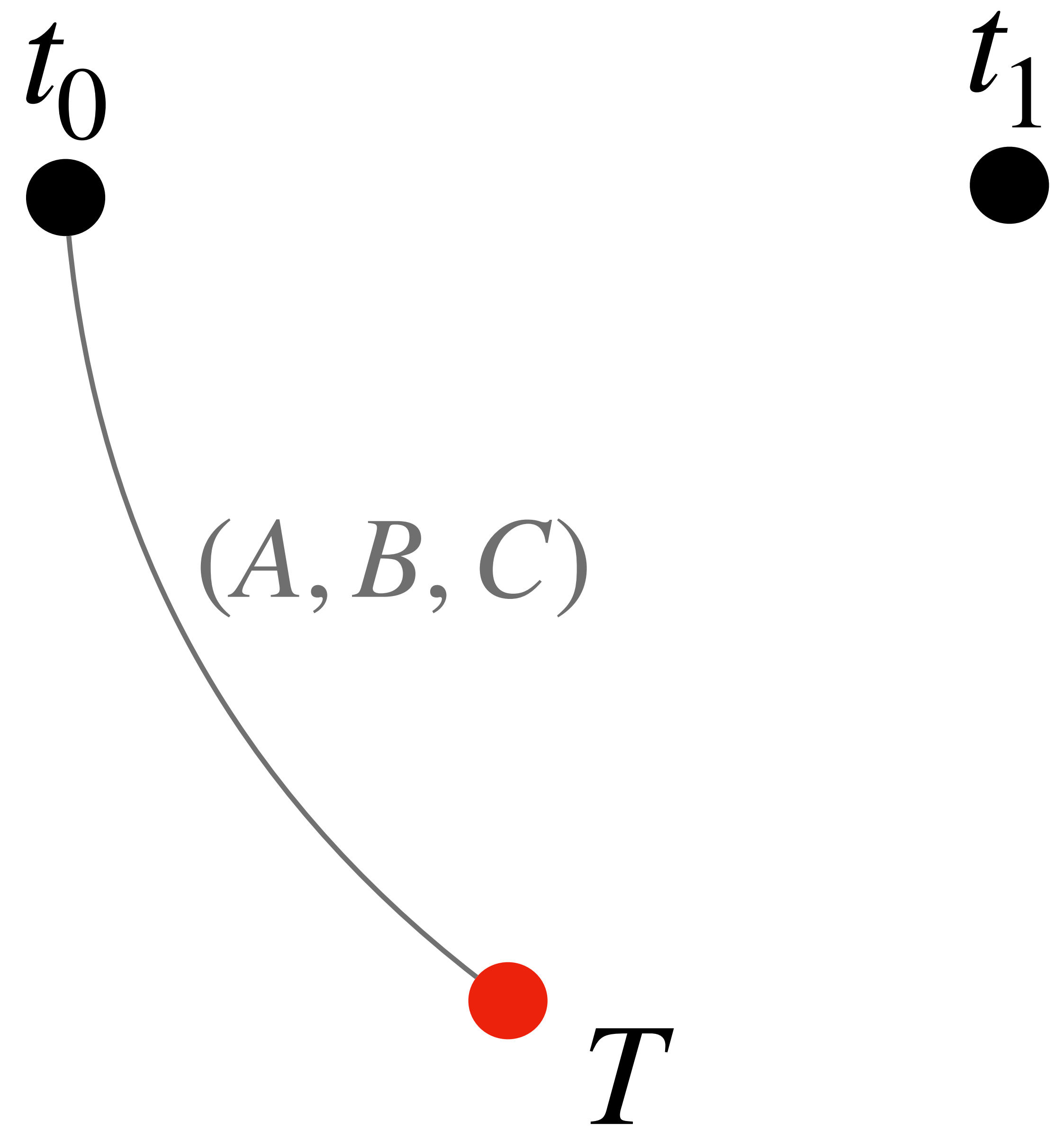
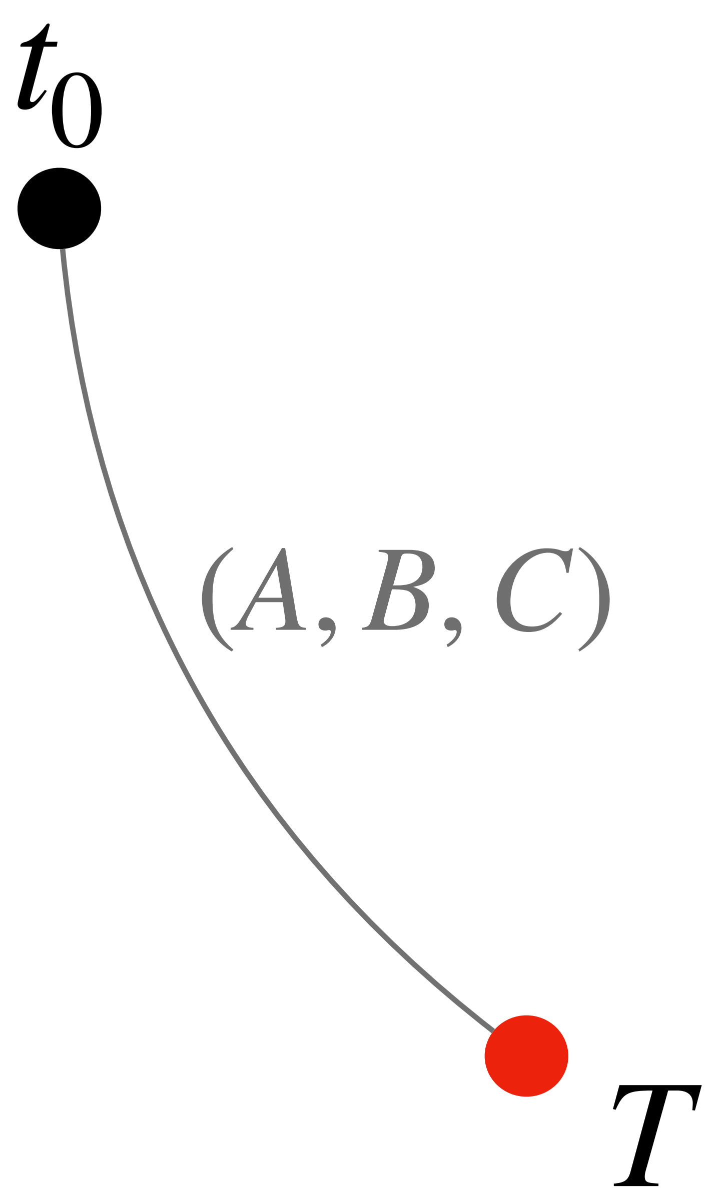


Is it **hiding**?

DFG paper



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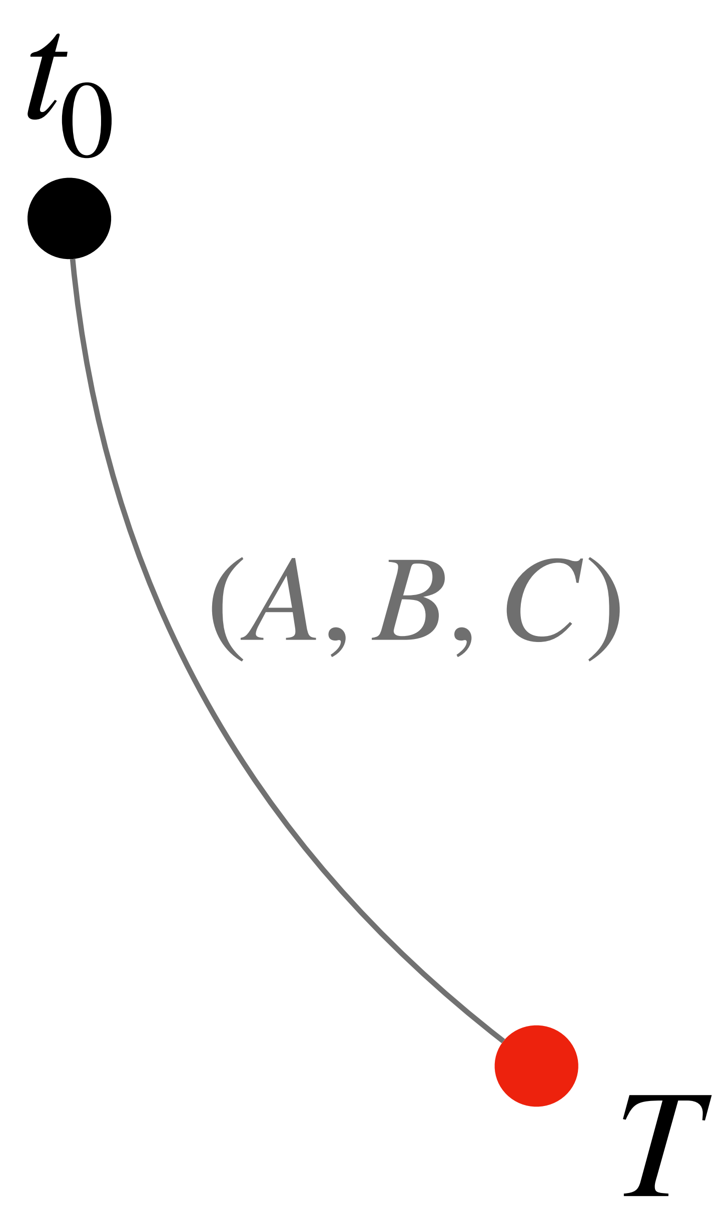

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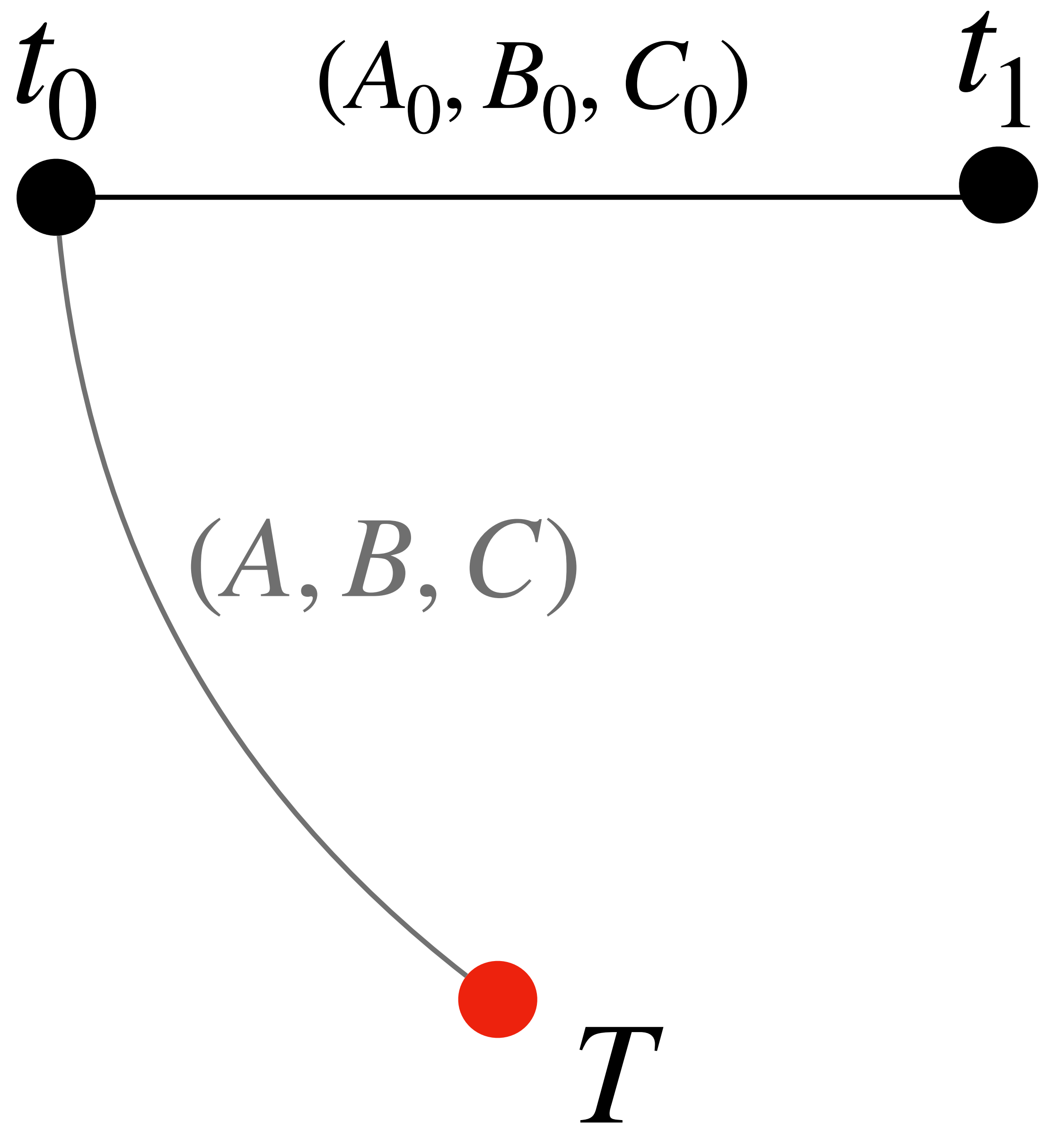

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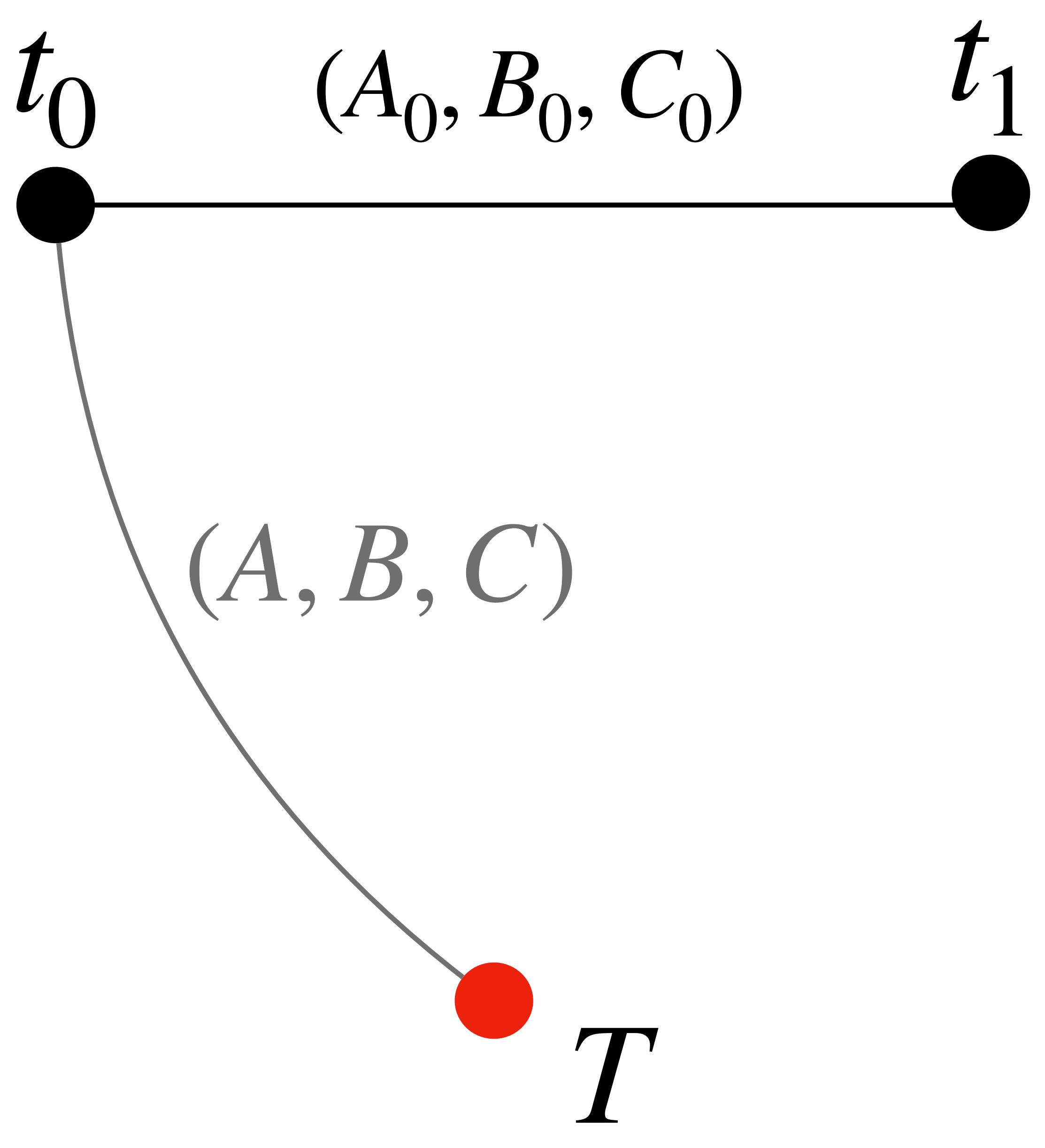

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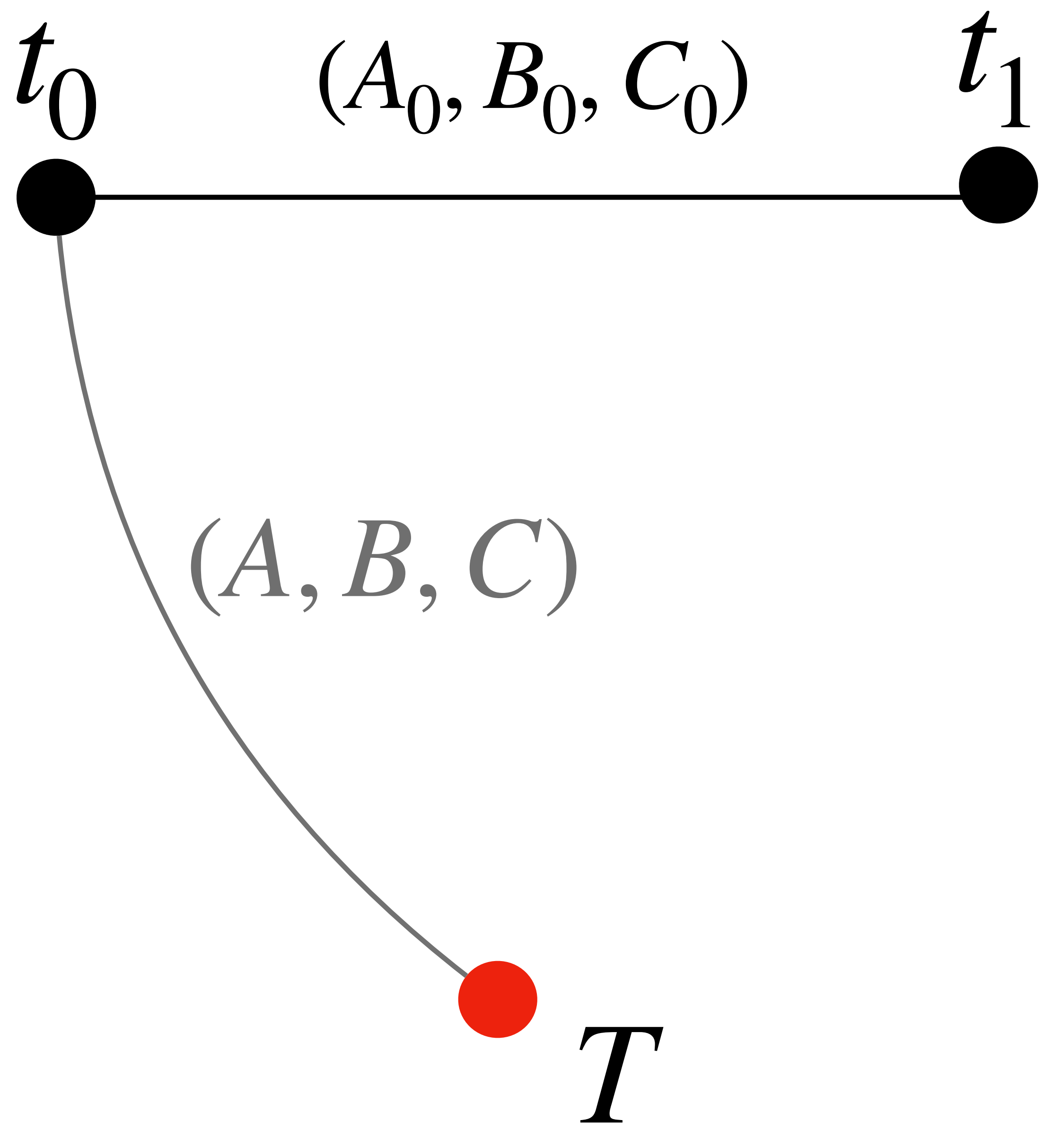

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$$T = (A, B, C) \star t_0$$
$$= (A, B, C) \star ((A_0, B_0, C_0) \star t_1)$$

Tensors

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rank-1 tensor : $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$

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hard problem :

given a tensor $T := [T_1, \ldots T_n]$,

compute the rank of T

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For **random** tensors, this problem is believed to be hard

DFG paper

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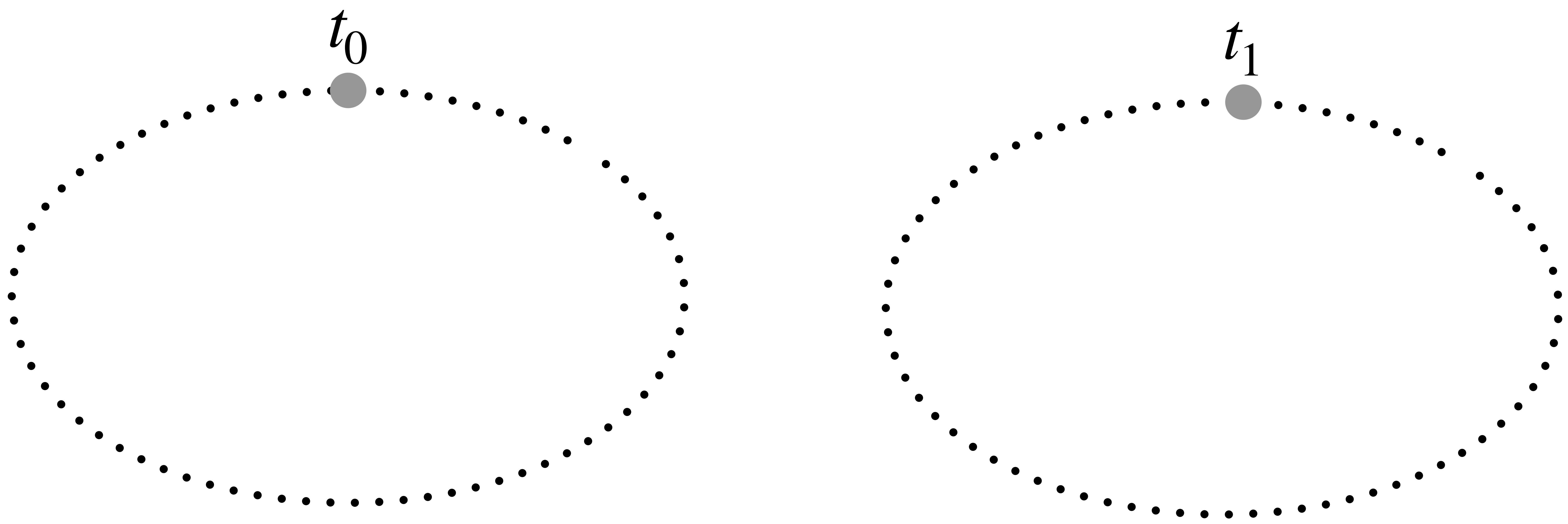
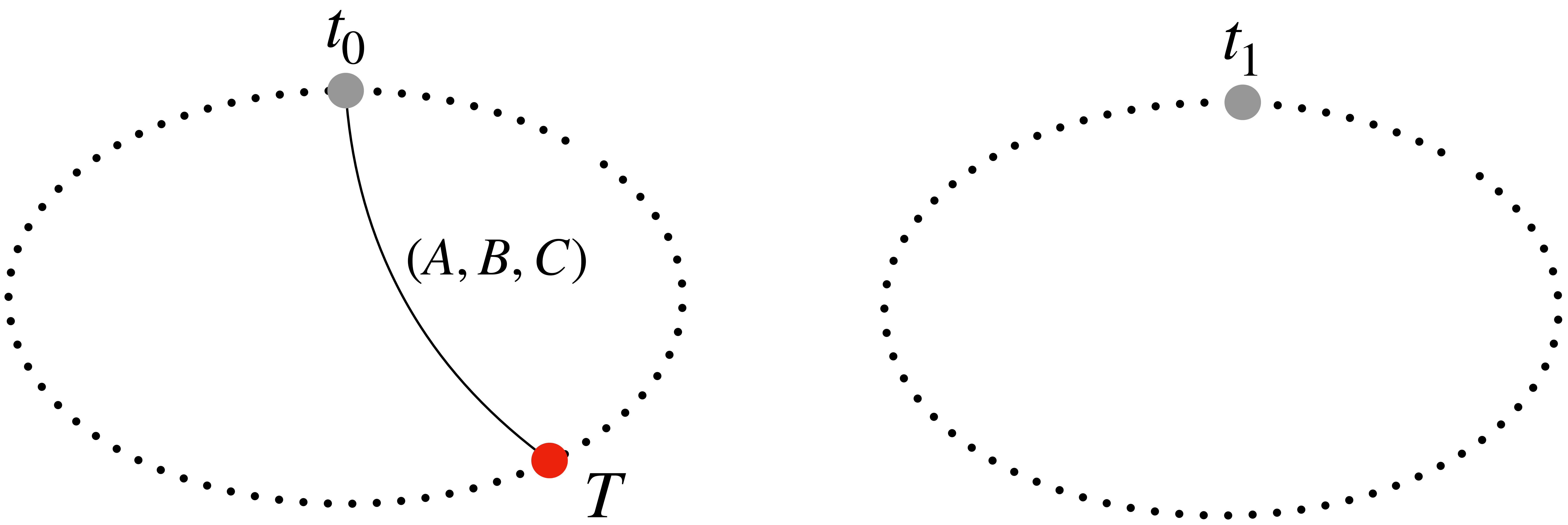


- we need t_0 and t_1 to be in different orbits

DFG paper



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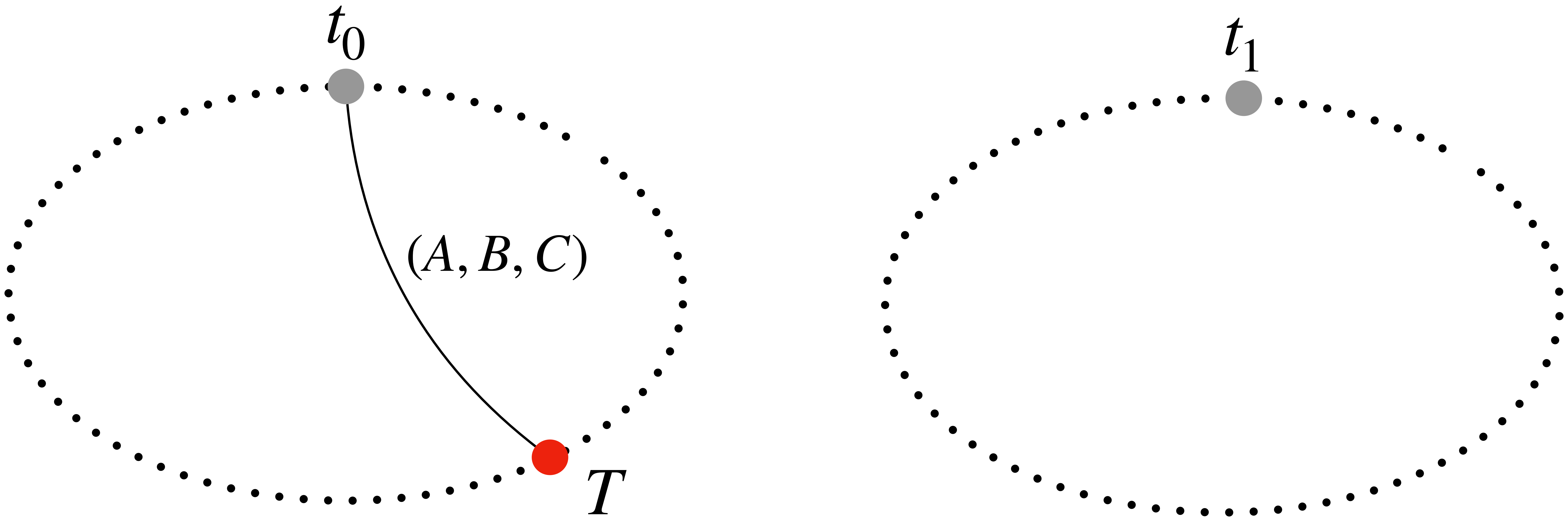

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Lemma : $rank((A, B, C) \star t) = rank(t)$

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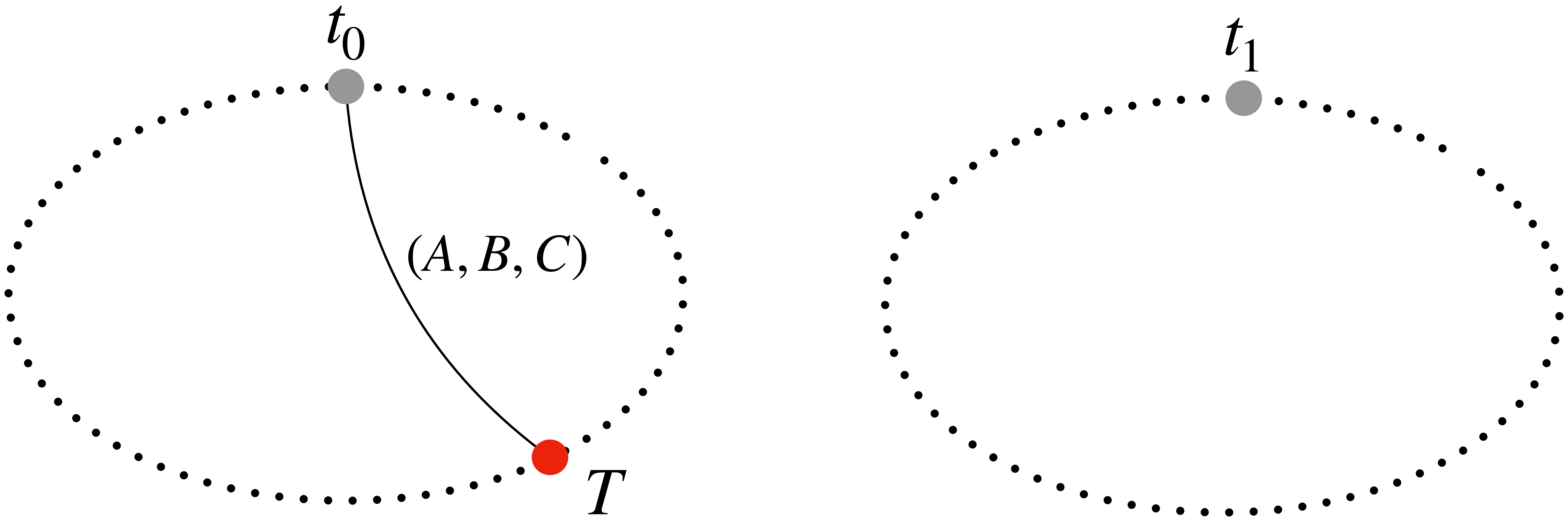

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$$t_0 = \sum_{i=1}^{3} e_i \otimes e_i \otimes e_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$t_1 = \sum_{i=1}^{2} e_i \otimes e_i \otimes e_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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An example, over $\mathbb{F}_7$:

$$\left(\begin{bmatrix}5&4&2\\4&2&0\\1&1&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&3&2\\0&2&4\end{bmatrix}, \begin{bmatrix}5&2&3\\2&1&1\\5&4&2\end{bmatrix} \right) \star t_1 = \begin{bmatrix}4&3&4\\3&5&6\\2&1&4\end{bmatrix}, \begin{bmatrix}6&1&6\\5&6&3\\1&4&2\end{bmatrix}, \begin{bmatrix}5&2&5\\6&3&5\\4&2&1\end{bmatrix}$$

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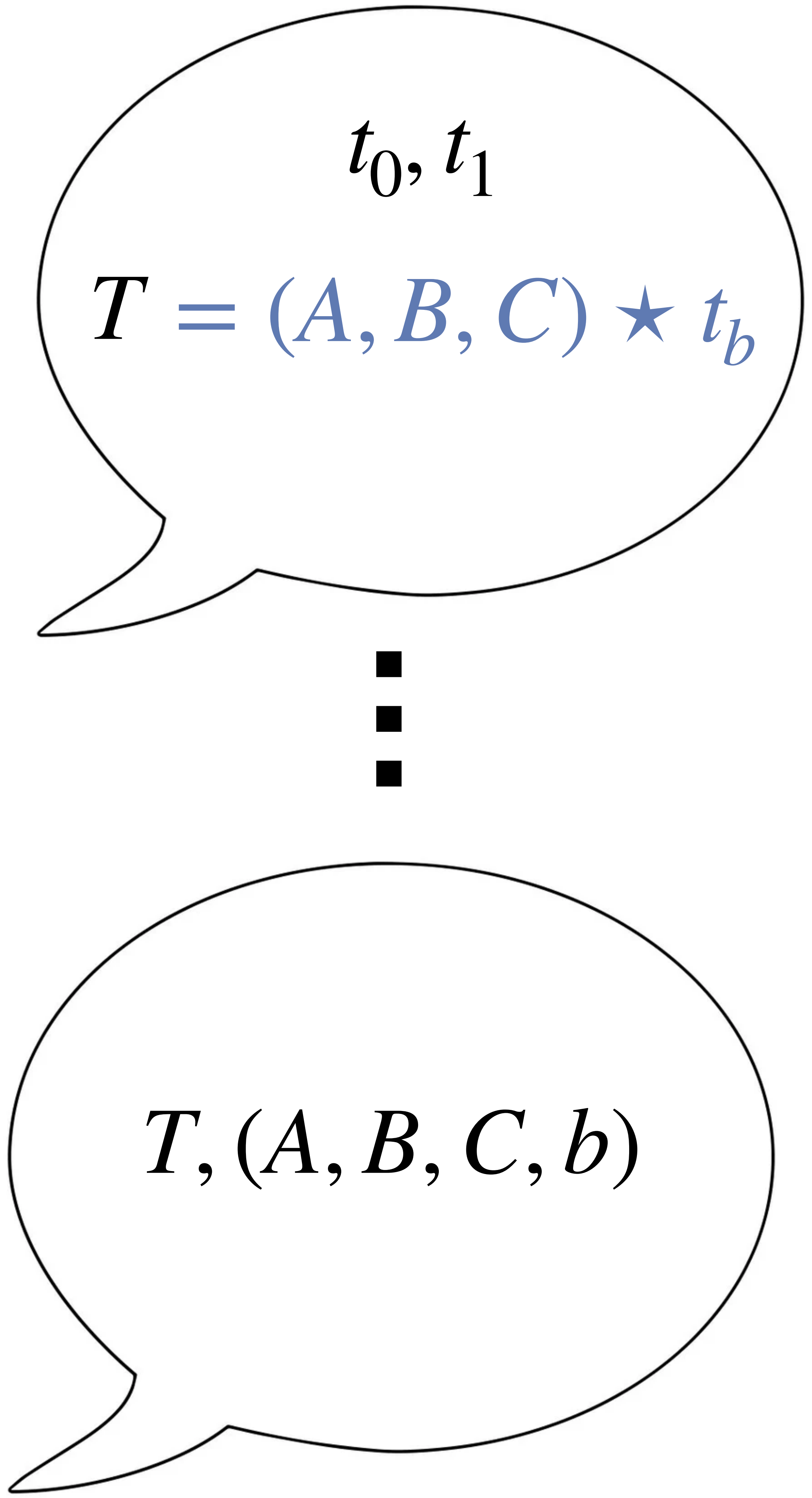

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$T, (A, B, C, b)$

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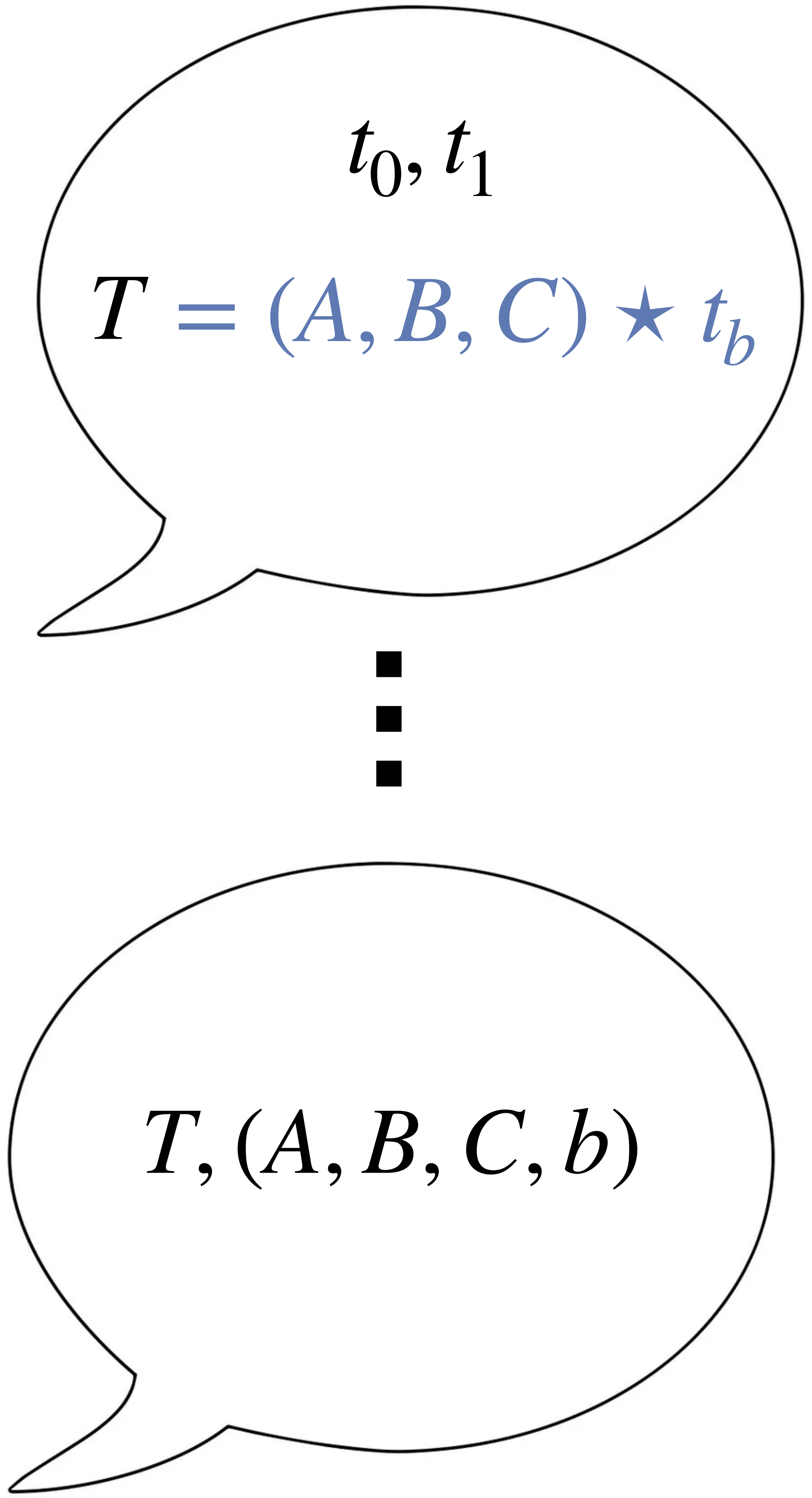

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Distinguishing attack

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We say the rank of $\mathbf{u} = (u_1, \ldots u_n)$ in $T = [T_1, \ldots T_n]$ is exactly

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i.e. points $\mathbf{u}$ such that $rank\left(u_1T_1 + \cdots u_nT_n\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Distinguishing attack

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Recall that

$$t_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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t_0 only has the trivial rank 0 point : $\mathbf{u} = (0,0,0)$

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$$t_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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t_1 has one (non-trivial) rank 0 point : scalar multiples of $\mathbf{u} = (0,0,1)$

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So t_1 has one rank 0 point and t_0 has no rank 0 points

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Thus, given some commitment, $T = (A, B, C) \star t_b$,

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→ $O(n^4)$ complexity

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Parameters

n	q
14	4093
22	4093
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$\rightarrow$ Runtime < 1 second

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This attack broke **hiding** and a special case of dTIP

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What about cTIP?

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A first attempt : Gröbner basis?

We can use a *Gröbner basis* to solve systems of multivariate polynomials

→ uses *Buchberger's algorithm*

→ manipulates the polynomials to eventually apply Gaussian elimination

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Upon first try, our instance has too many solutions…

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$$\begin{bmatrix} \lambda_a & 0 & 0 \\ 0 & \lambda_a & 0 \\ 0 & 0 & \lambda_a \end{bmatrix} \cdot \begin{bmatrix} \lambda_b & 0 & 0 \\ 0 & \lambda_b & 0 \\ 0 & 0 & \lambda_b \end{bmatrix} \cdot \begin{bmatrix} \lambda_c & 0 & 0 \\ 0 & \lambda_c & 0 \\ 0 & 0 & \lambda_c \end{bmatrix} = I$$

Computational attack

How can we reduce the number of solutions?

The *stabilizer group* of t_0 is matrix triples (M_1, M_2, M_3) such that

$$(M_1, M_2, M_3) \star t_0 = t_0$$

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any permutation matrices (P, P, P),

e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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➡ $O(n^6)$ complexity

Computational attack

Runtime for the attack:

n	q	Time (s)
14	4093	9.3
22	4093	141.6
30	2039	858.9

MinRank

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There was only 1 rank-0 matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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MinRank:

Given an integer $r \in \mathbb{N}$ and k matrices $M_1, \ldots M_k$,
find integers $x_1, \ldots x_k$ (not all zero) such that

$$rank\left(x_1 M_1 + \ldots + x_k M_k\right) \leq r$$

MinRank

We used the following work (*AsiaCrypt 2020*):

Improvements of Algebraic Attacks for solving the Rank Decoding and MinRank problems

Magali Bardet[4,5], Maxime Bros[1], Daniel Cabarcas[6], Philippe Gaborit[1], Ray Perlner[2], Daniel Smith-Tone[2,3], Jean-Pierre Tillich[4], and Javier Verbel[6]

[1] Univ. Limoges, CNRS, XLIM, UMR 7252, F-87000 Limoges, France
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[2] National Institute of Standards and Technology, USA
[3] University of Louisville, USA
[4] Inria, 2 rue Simone Iff, 75012 Paris, France
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Abstract. In this paper, we show how to significantly improve algebraic techniques for solving the MinRank problem, which is ubiquitous in multivariate and rank metric code based cryptography. In the case of

→ we were able to solve with **direct linearization**

→ for $r > 1$ the complexity quickly increases

Repair

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What do we need?

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→ tensors in different orbits

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→ no low rank points

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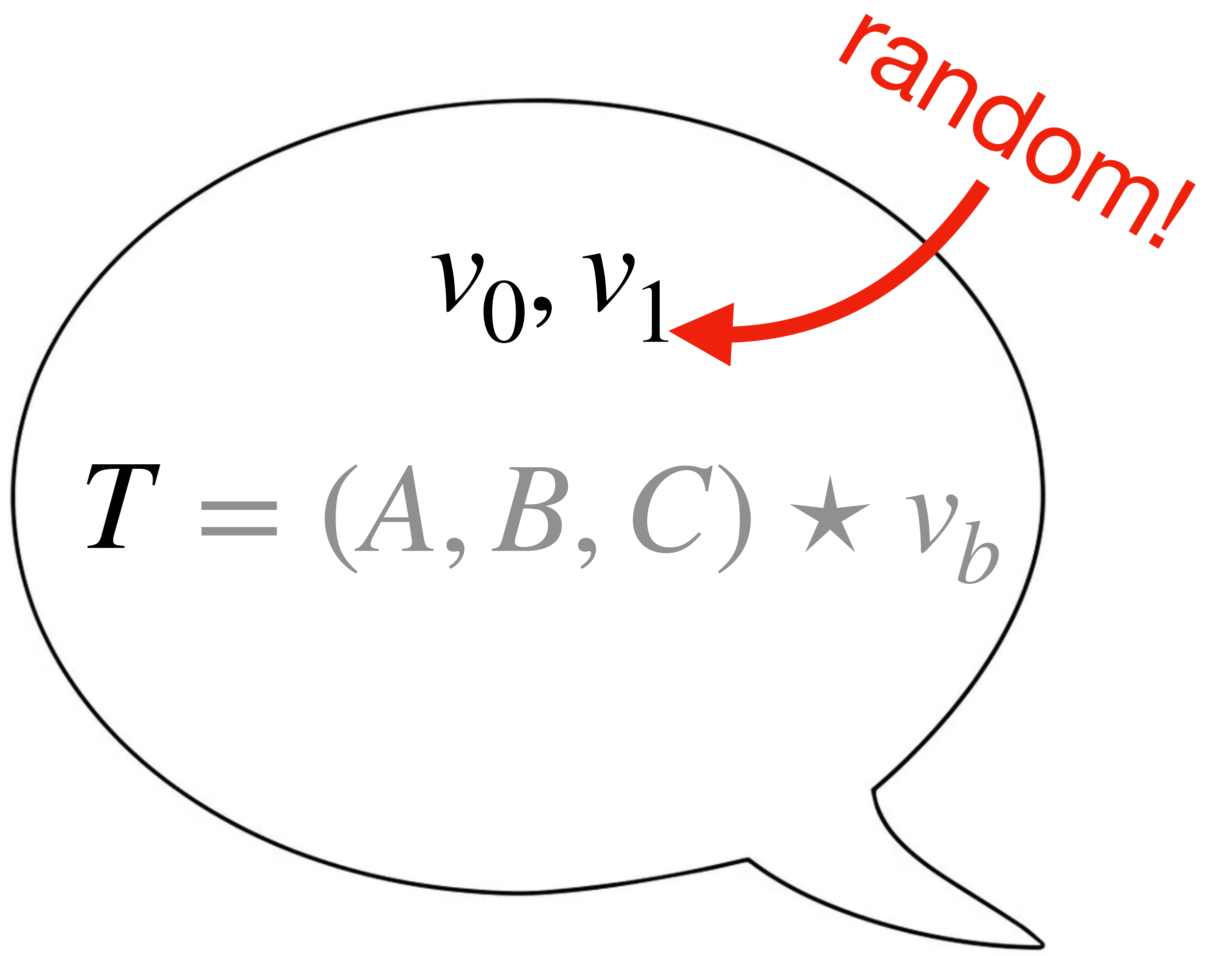

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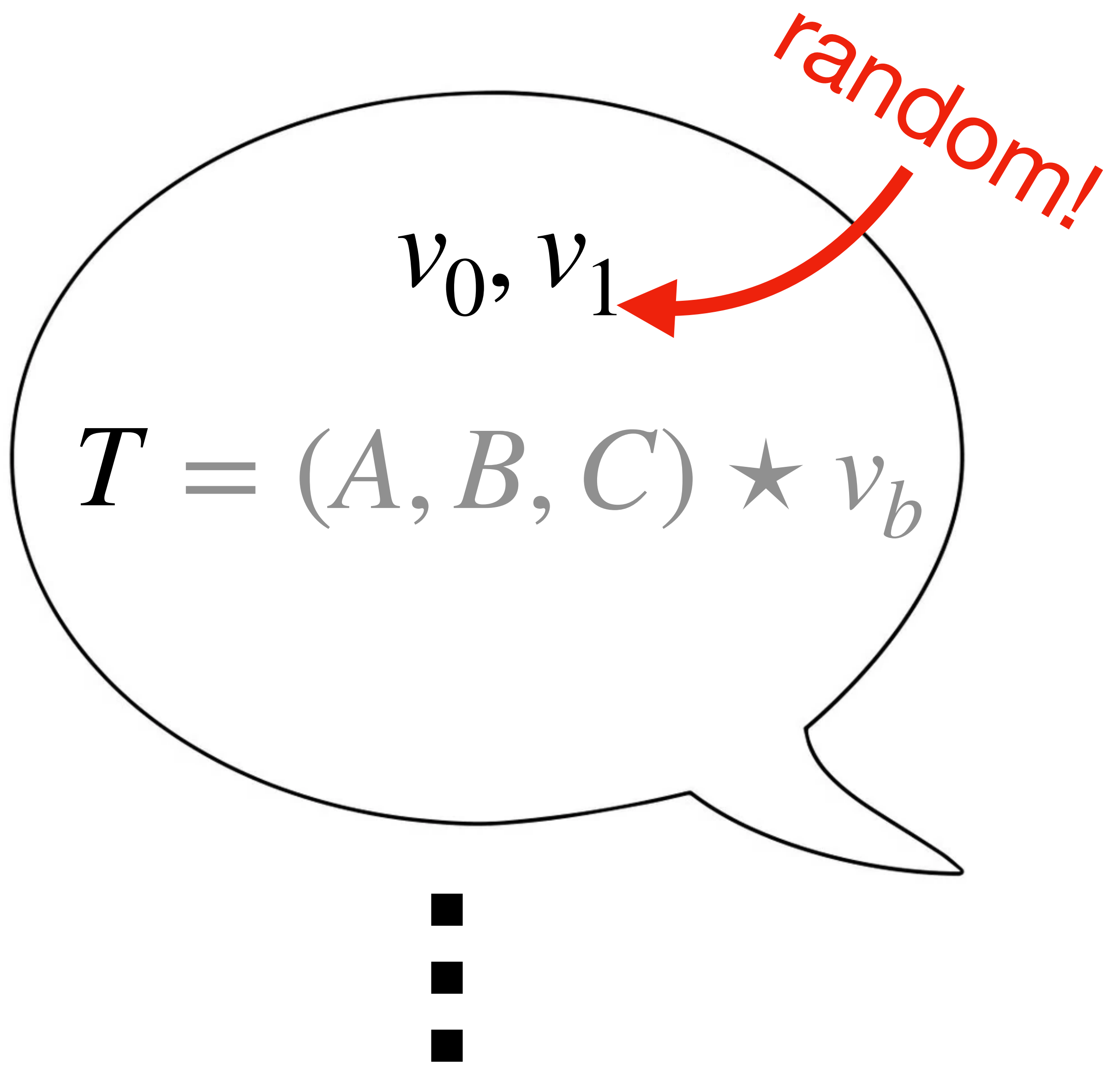

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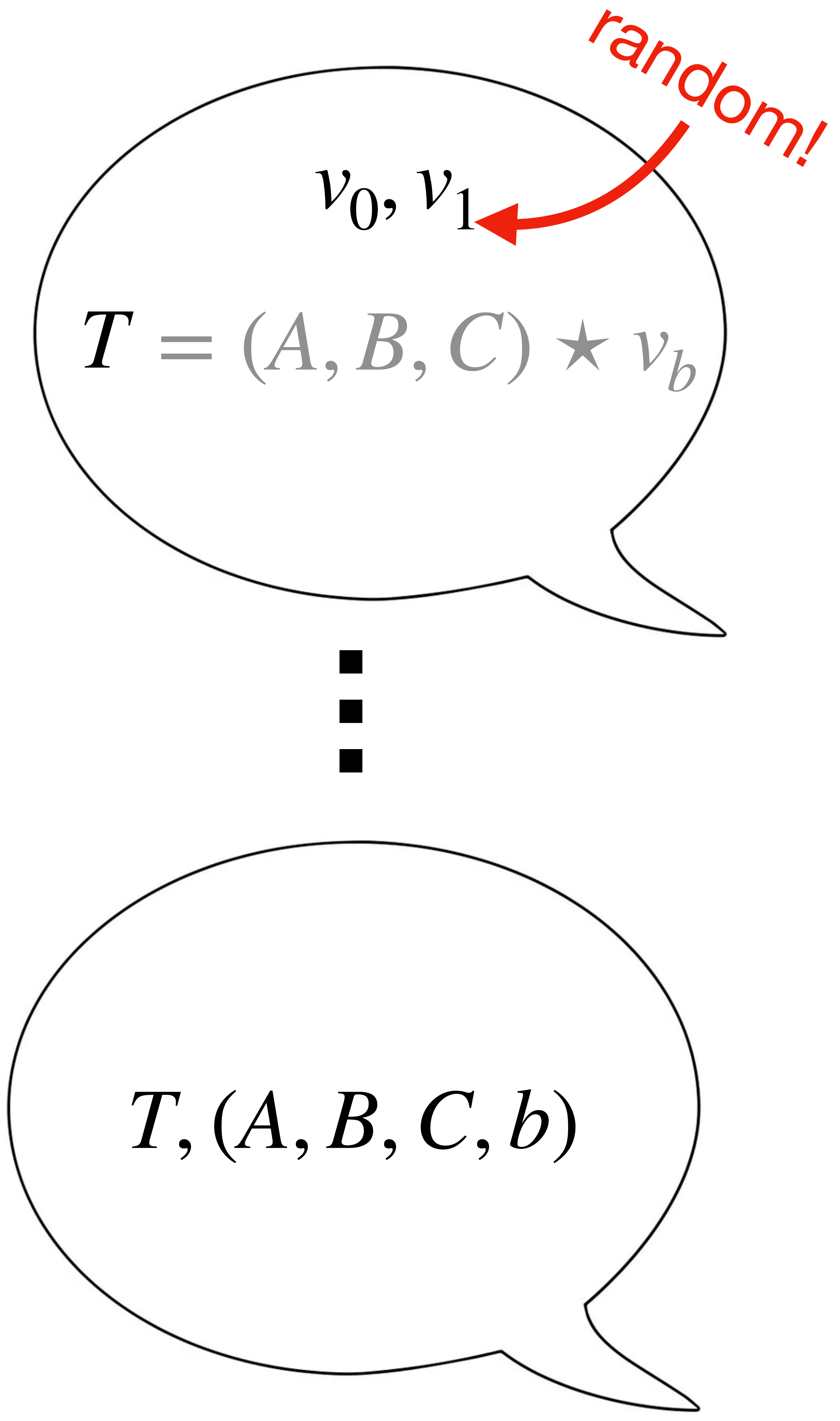

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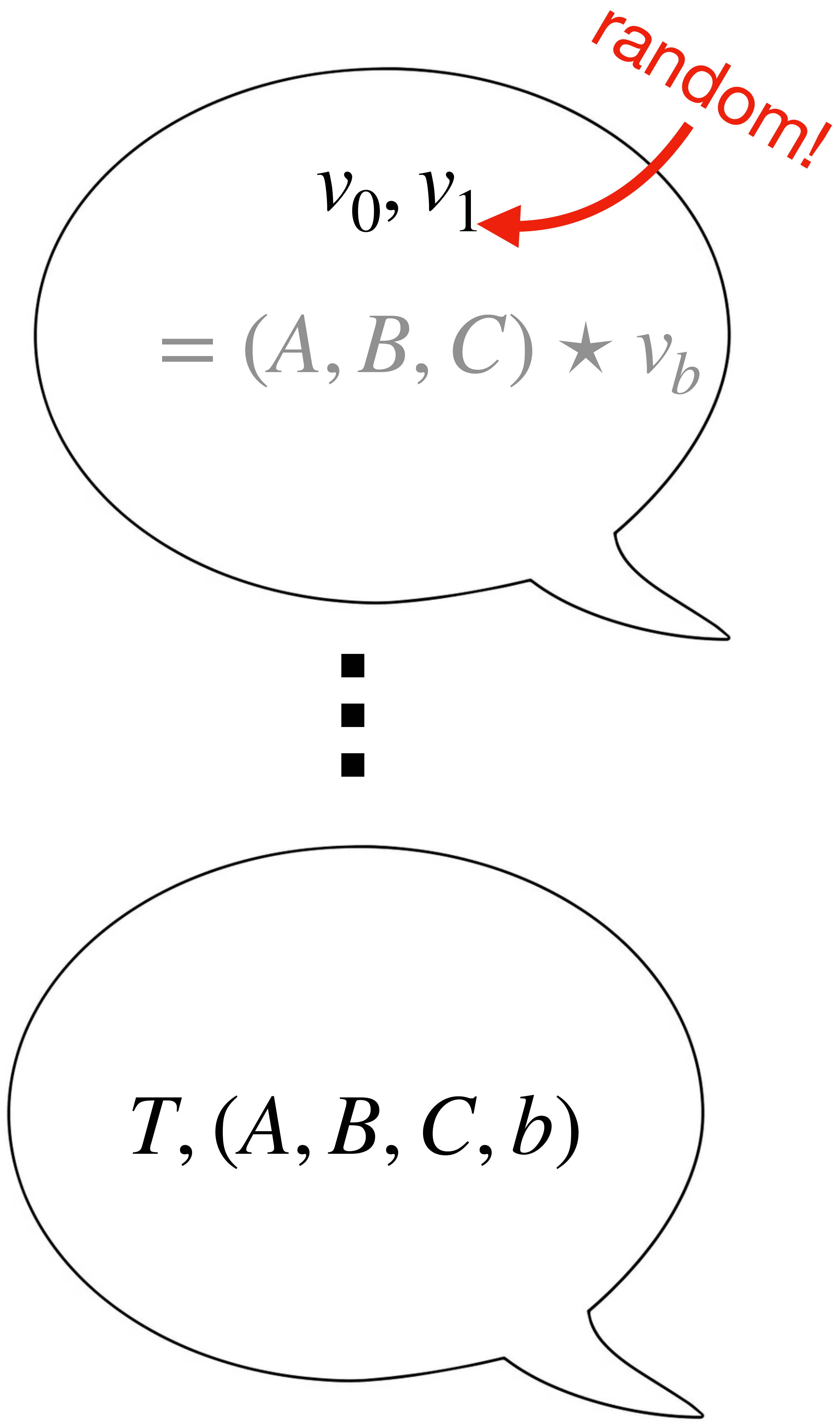
How can we ensure that v_0, v_1 are generated randomly?



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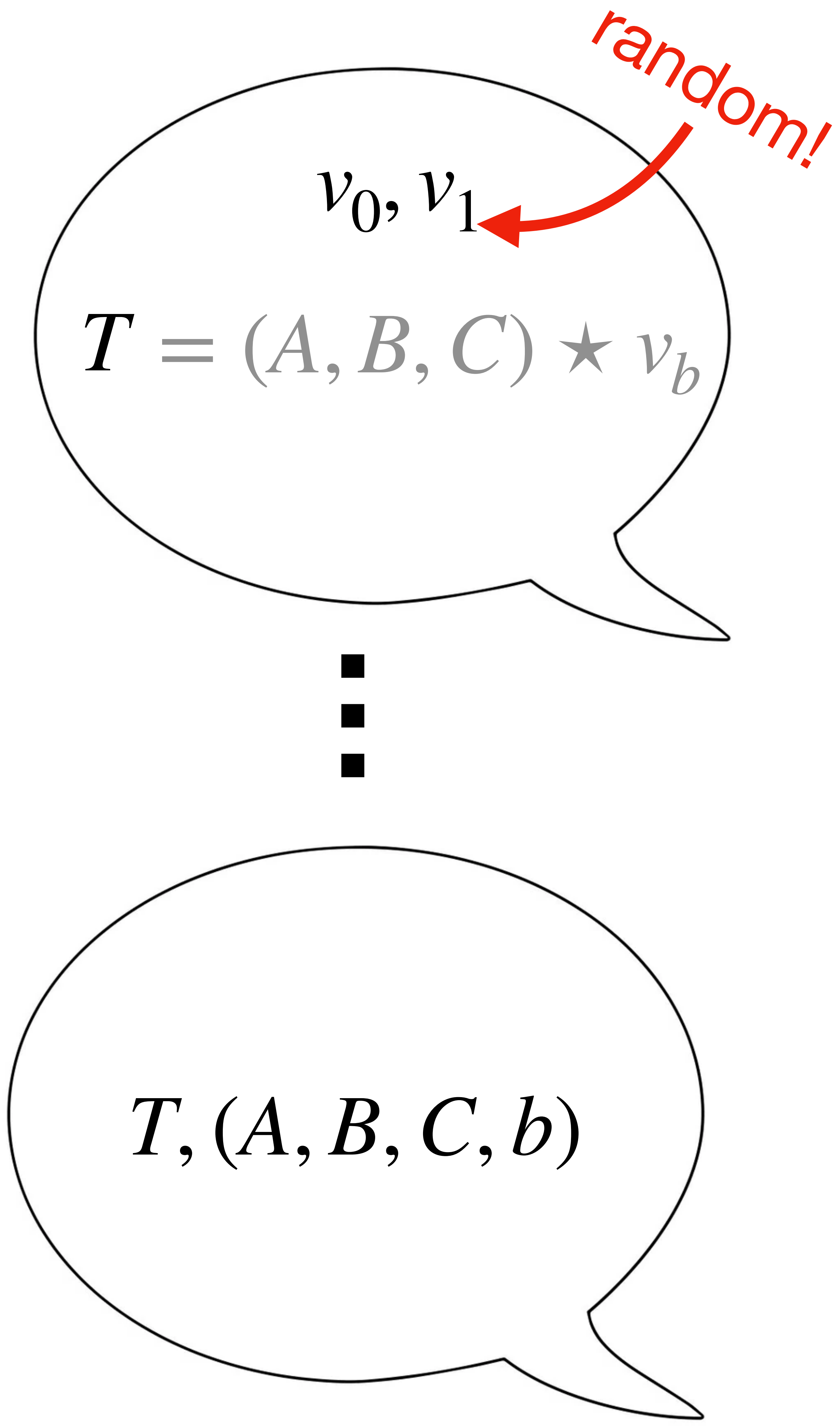

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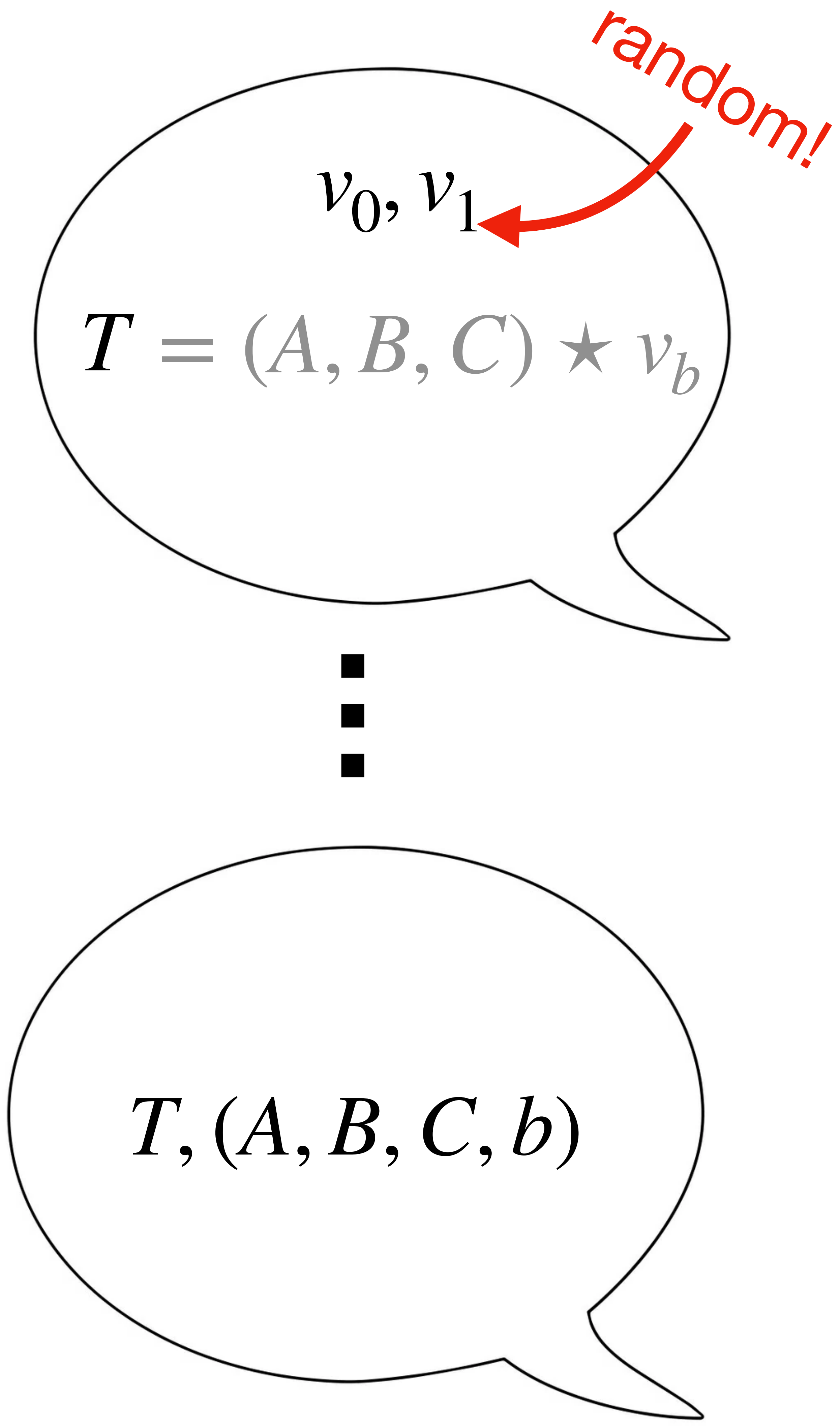
$\rightarrow$ pseudo-random number generator



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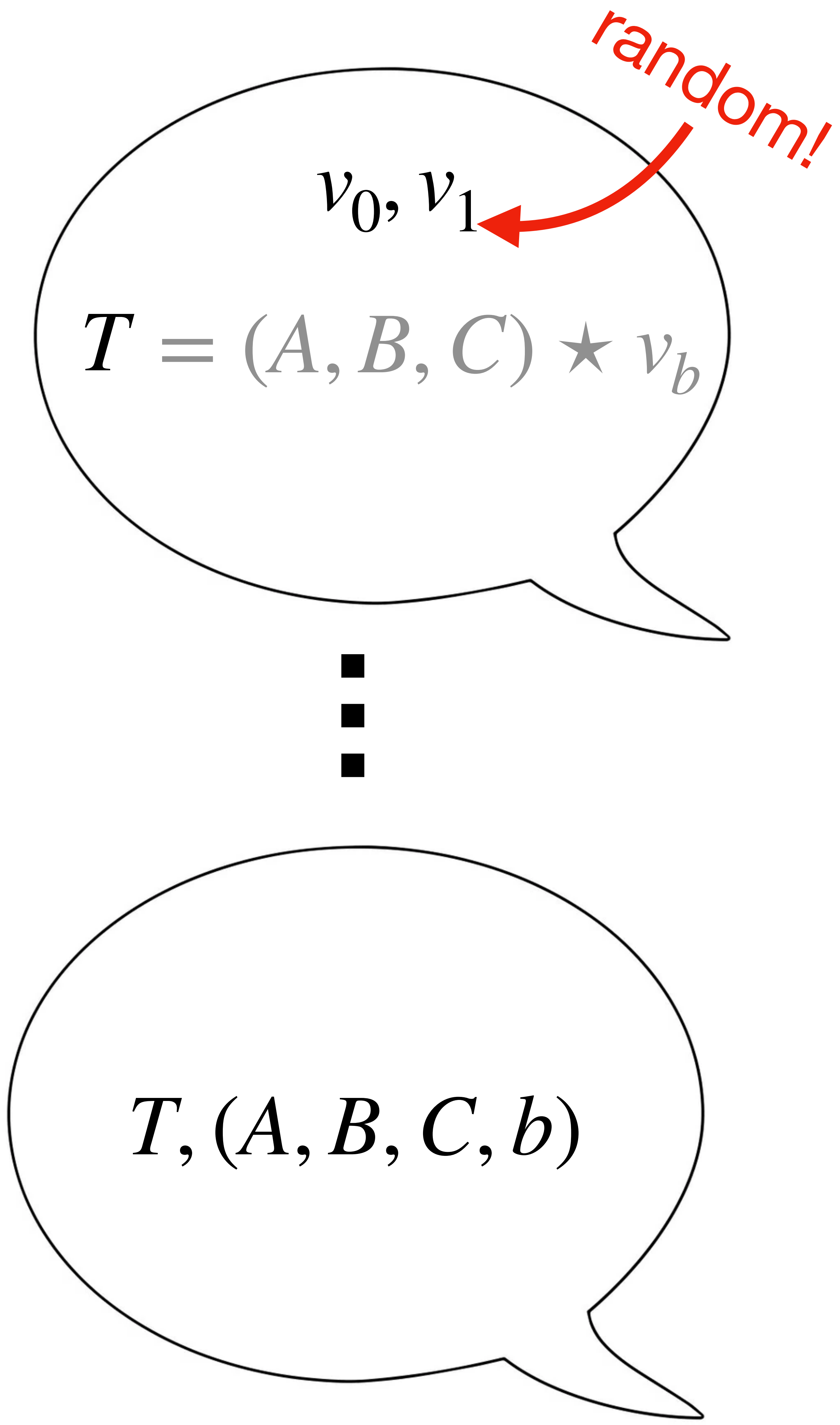
→ pseudo-random number generator

→ cryptographic hash function



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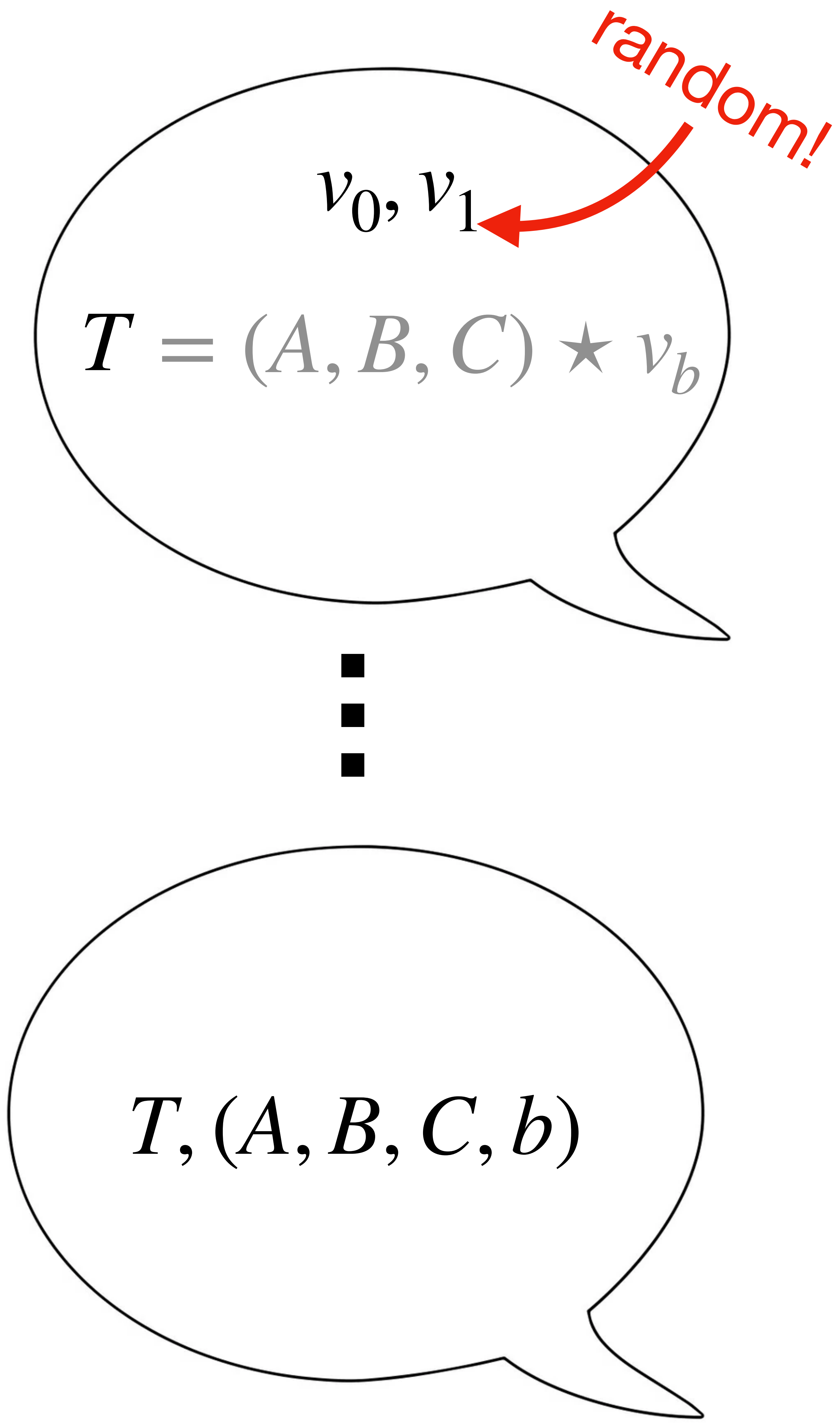
How can we ensure that v_0, v_1 are generated randomly?

→ pseudo-random number generator

→ cryptographic hash function

→ trusted third party



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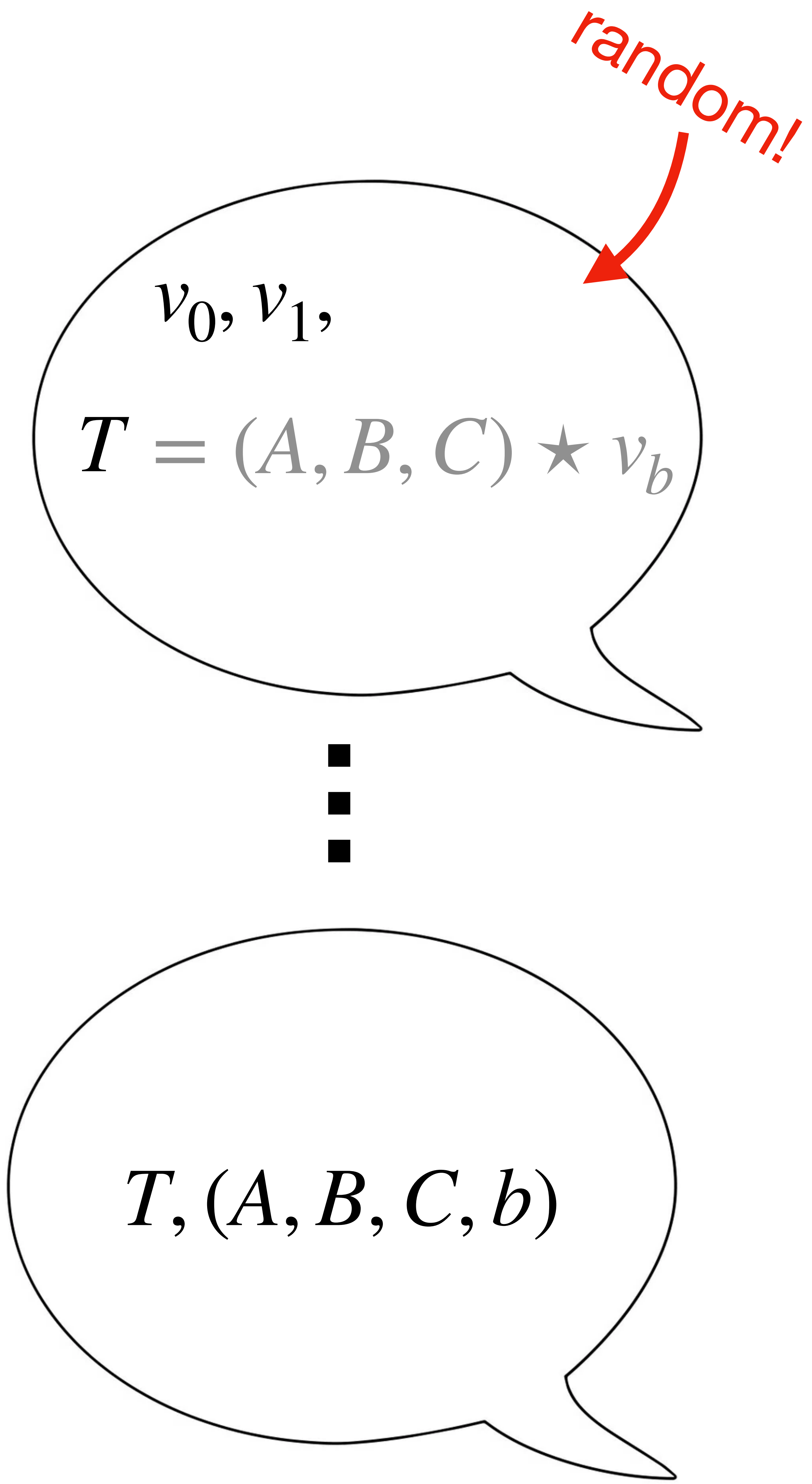

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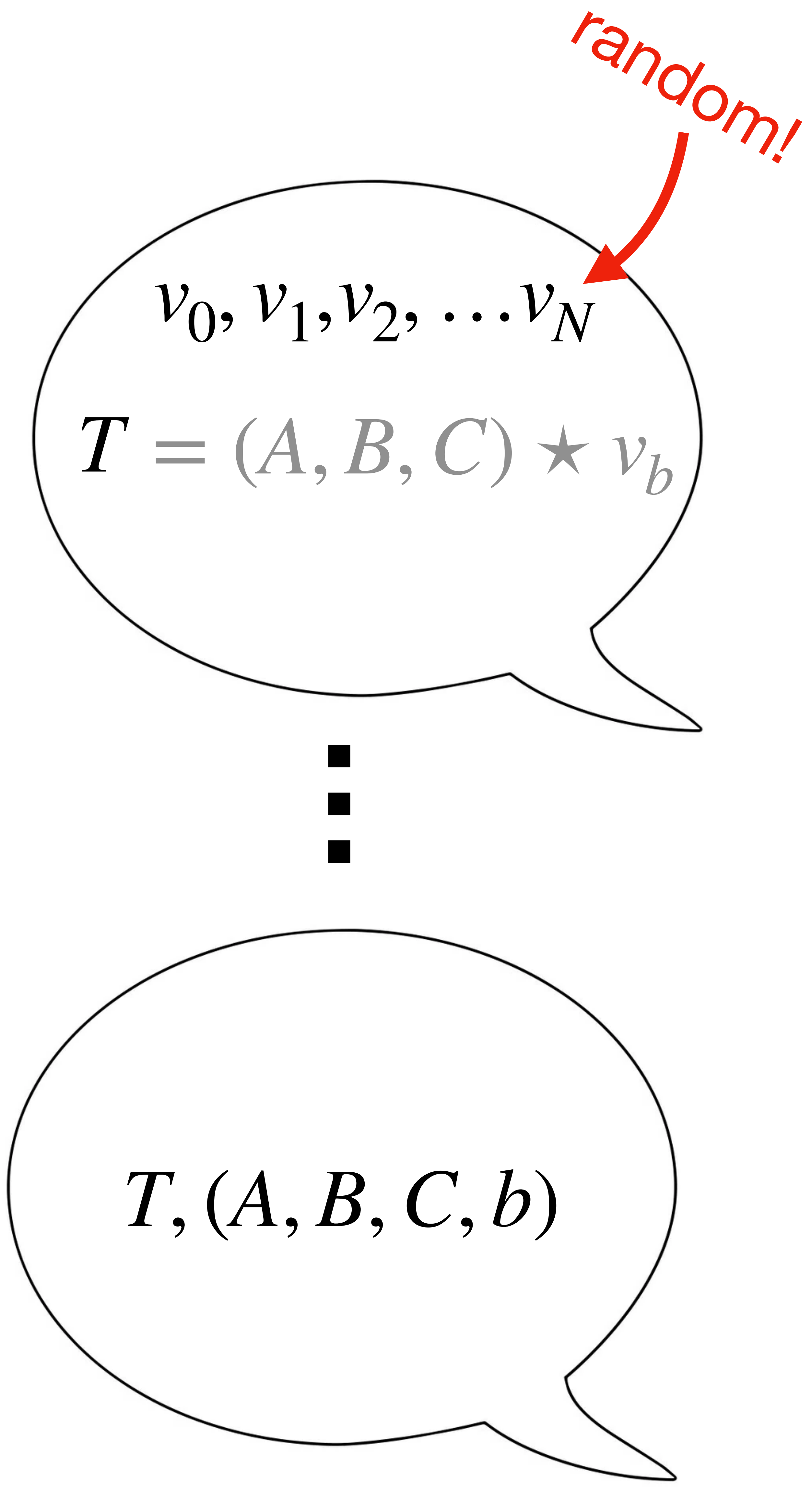

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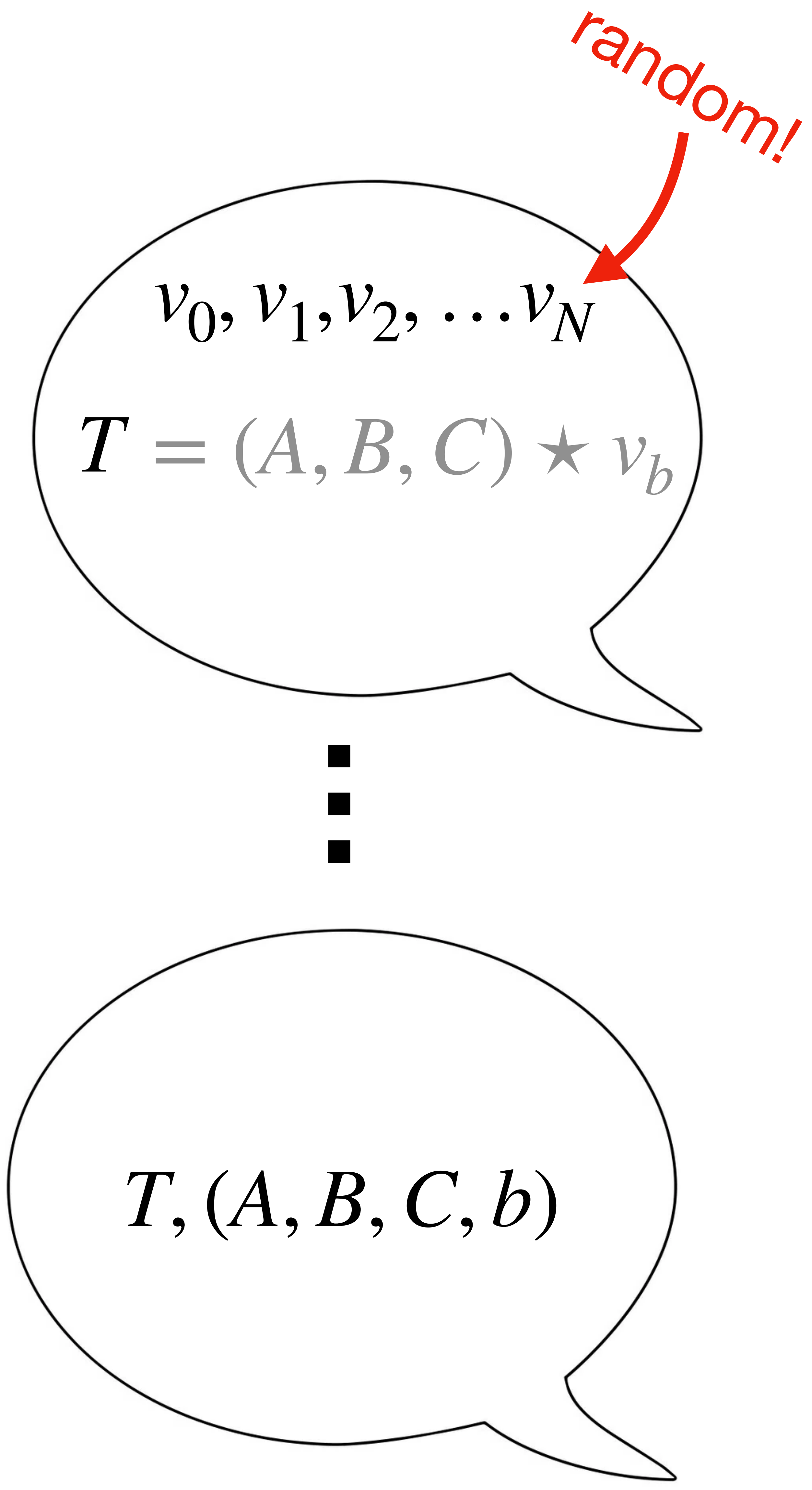

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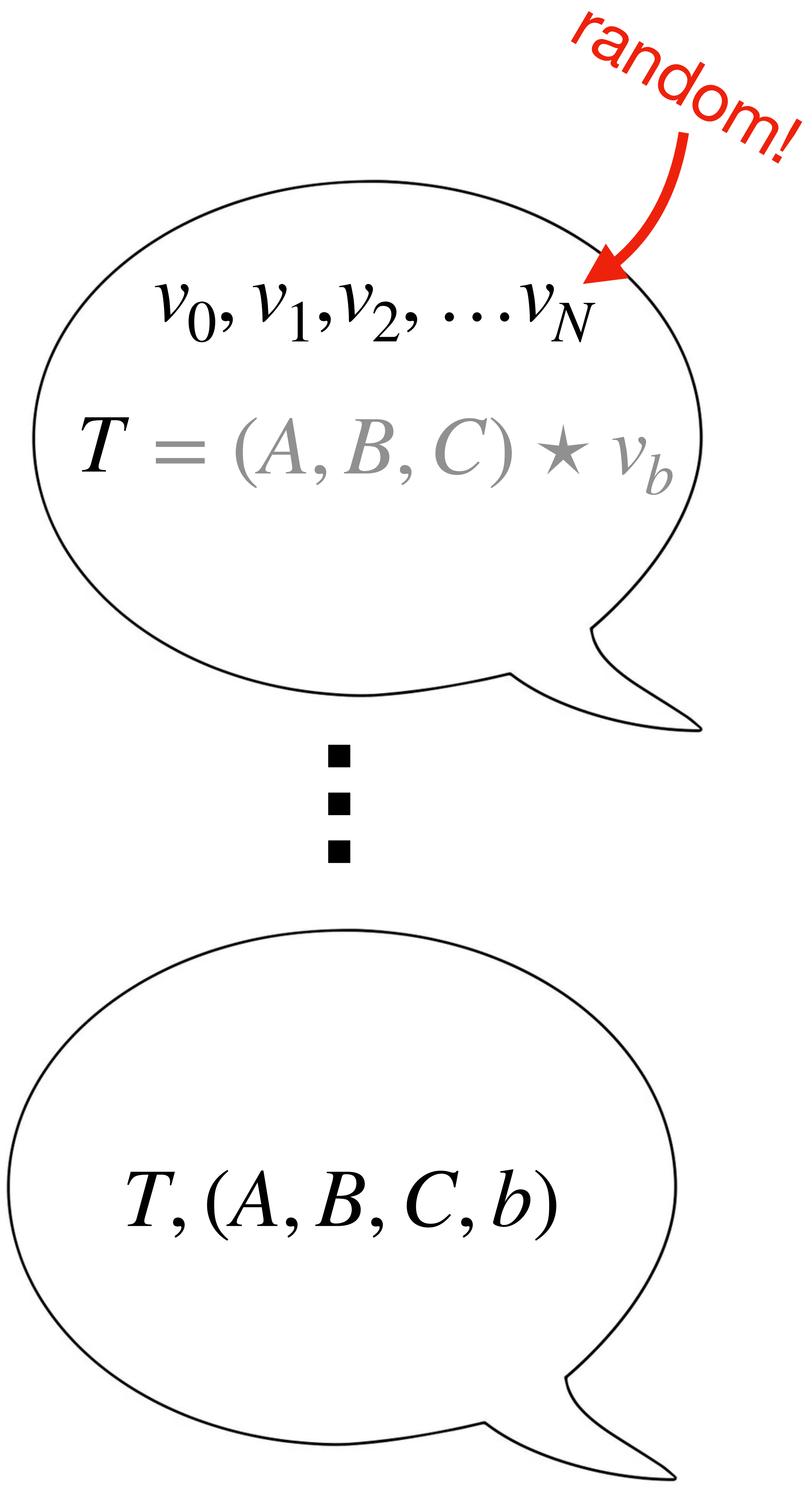

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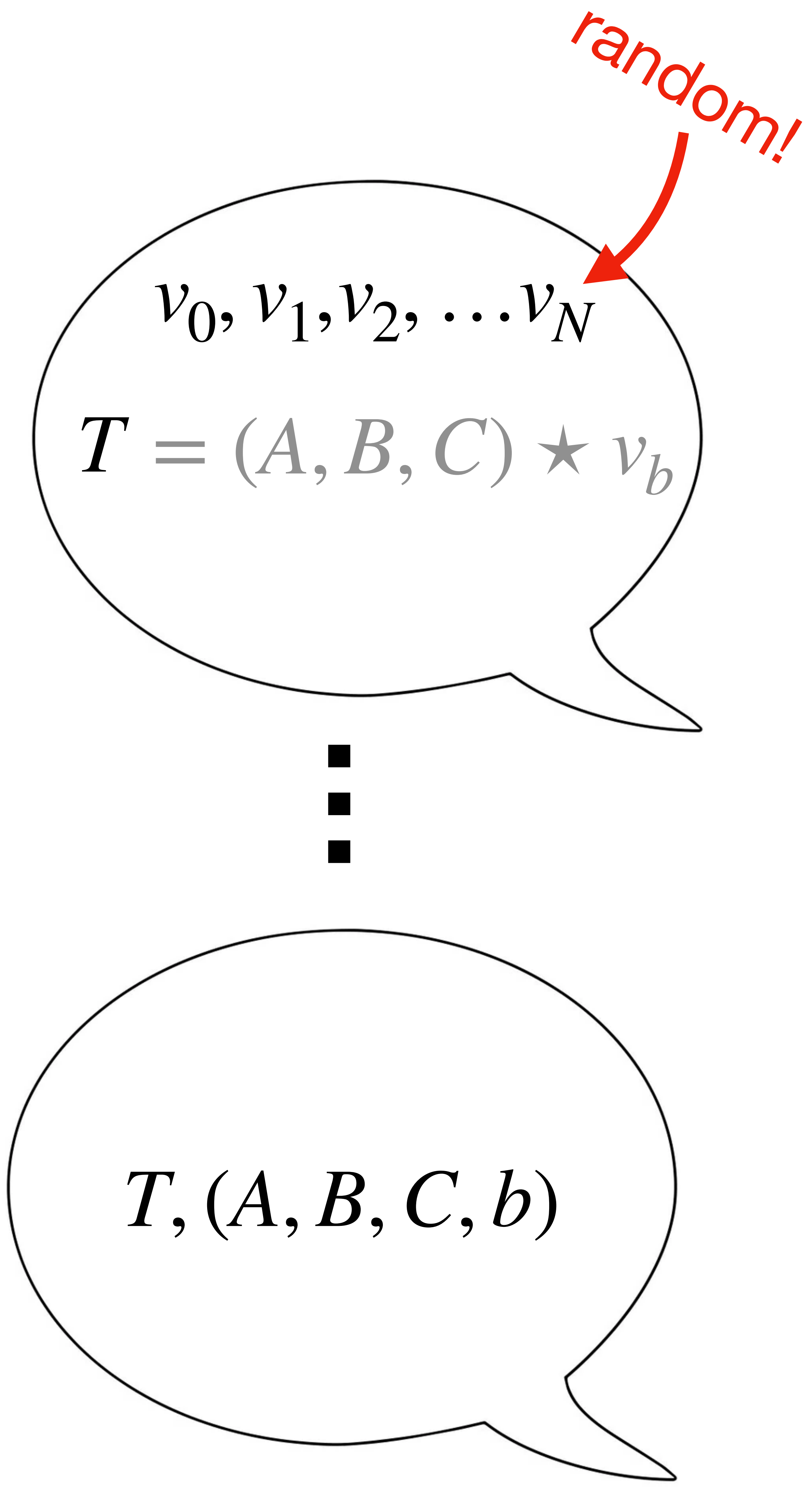
This gives a scheme that is

→ **statistically binding**

→ **computationally hiding**



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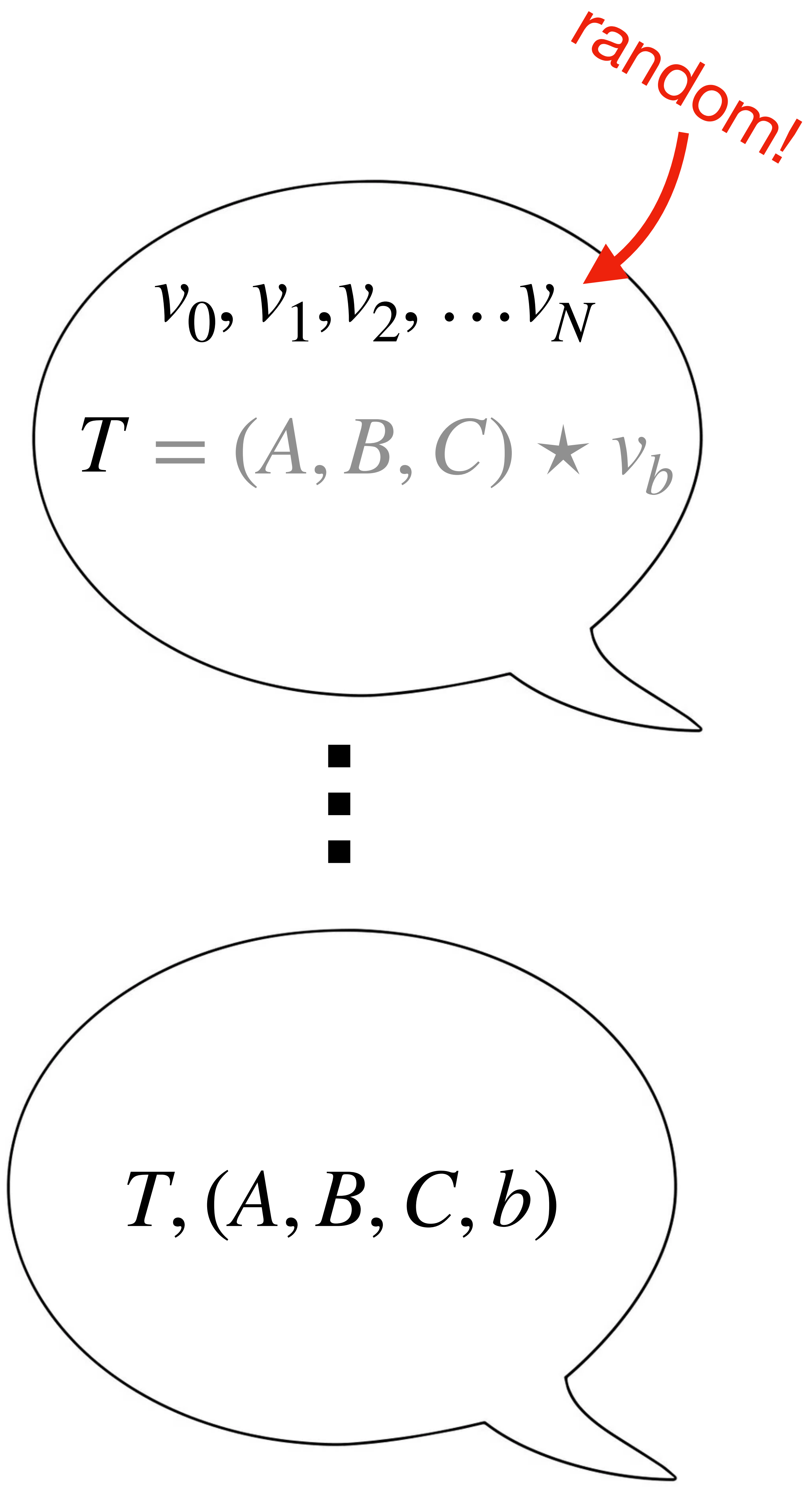
This gives a scheme that is

→ **statistically binding**

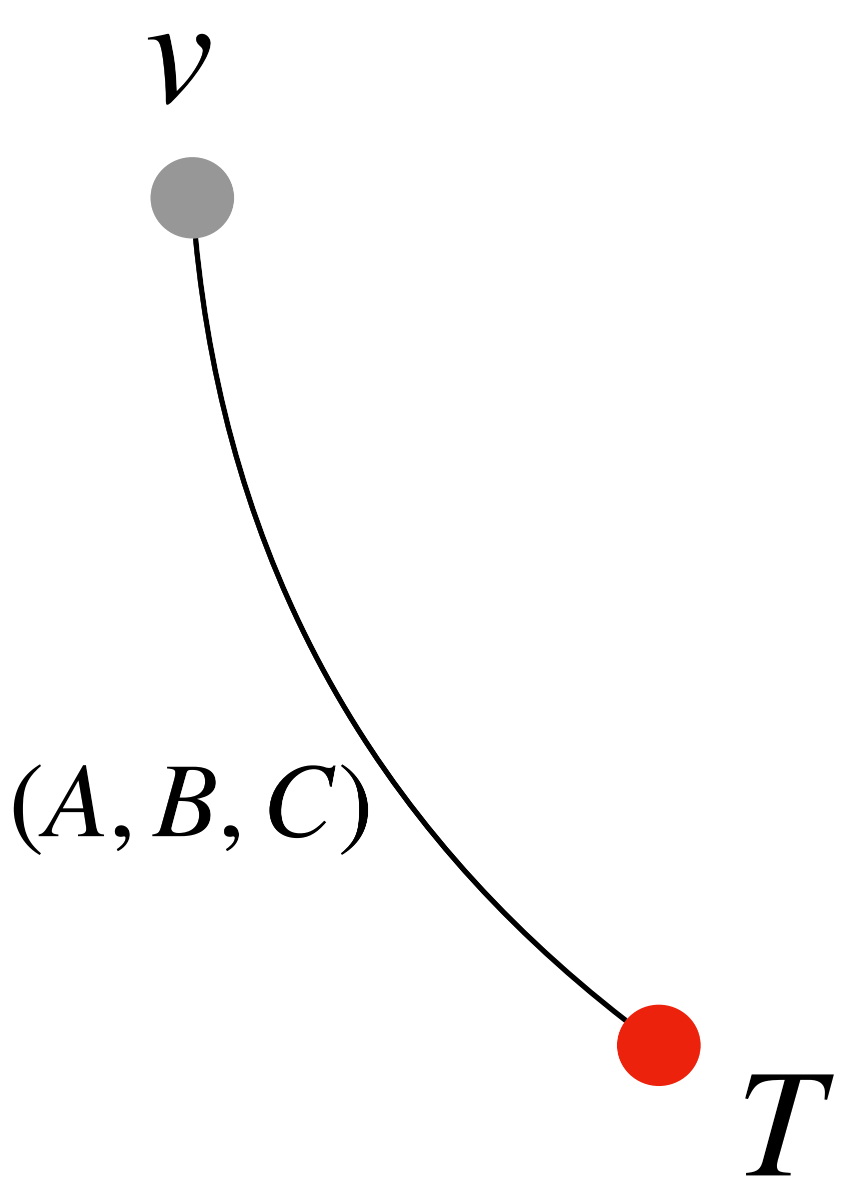
→ **computationally hiding**

→ complexity increases from $O(n^3)$ to $O(n^4)$



Proofs of knowledge

Sometimes we will want to prove that we know a committed value without revealing it



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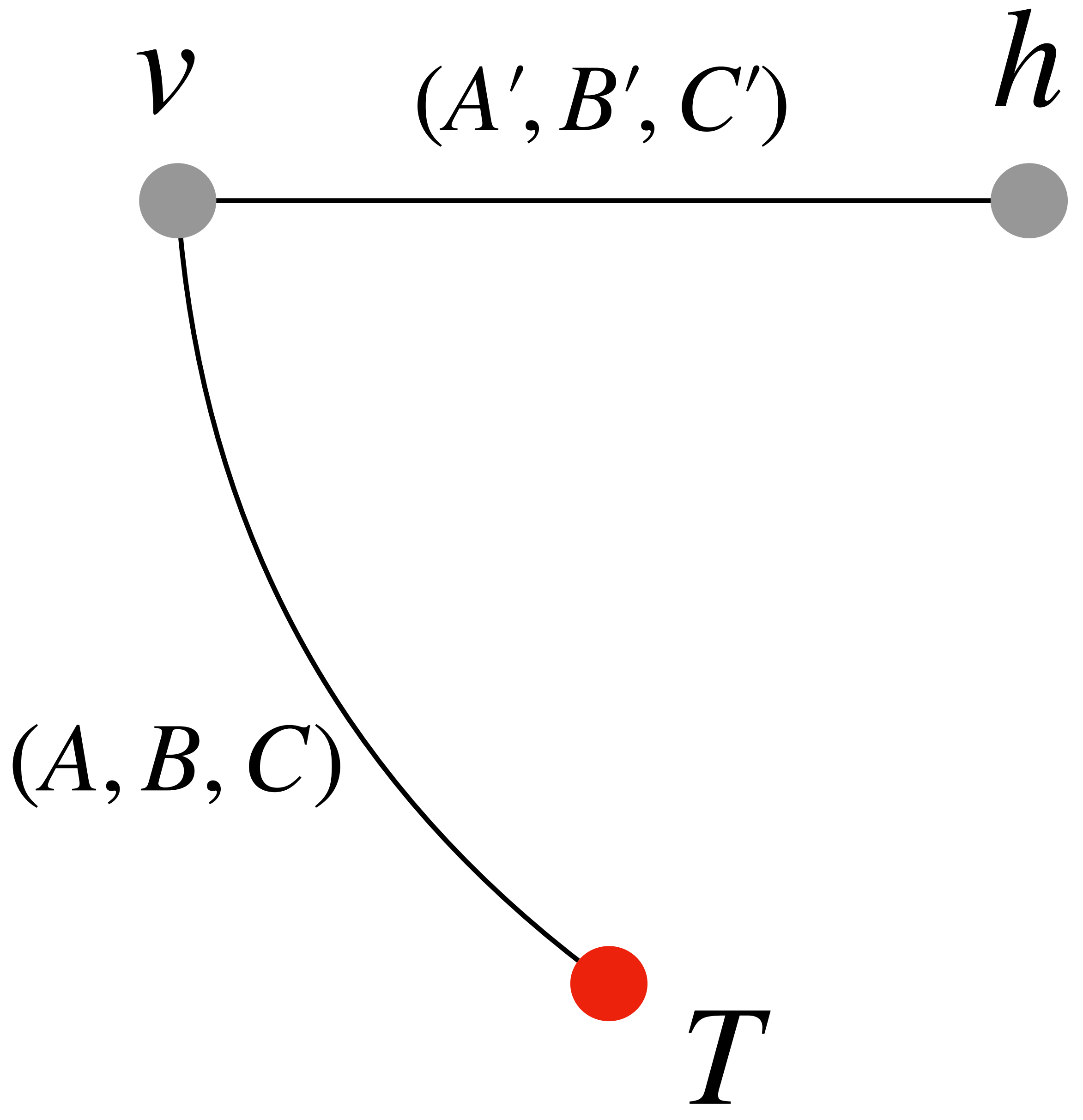

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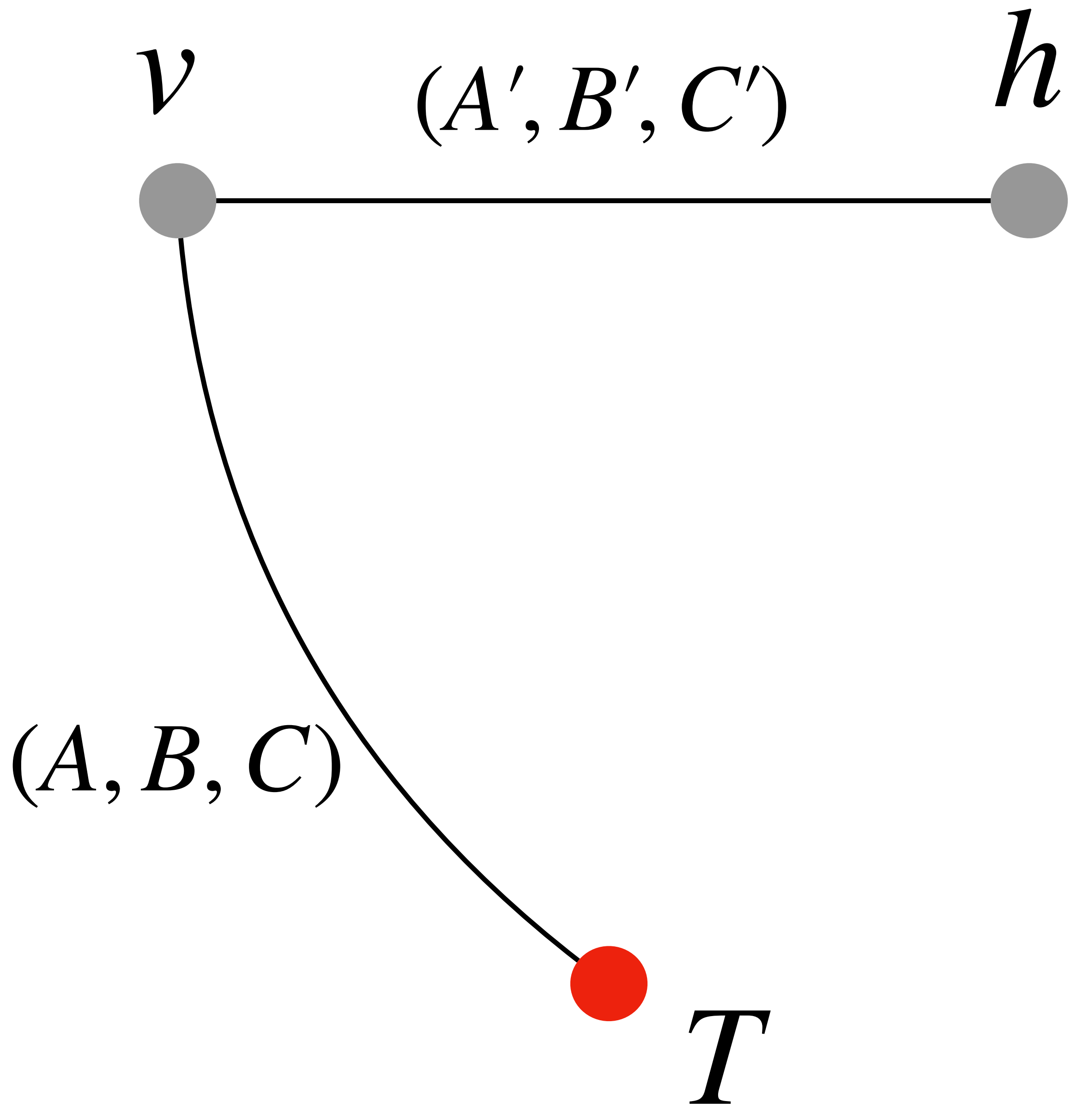

Prover:

Verifier:

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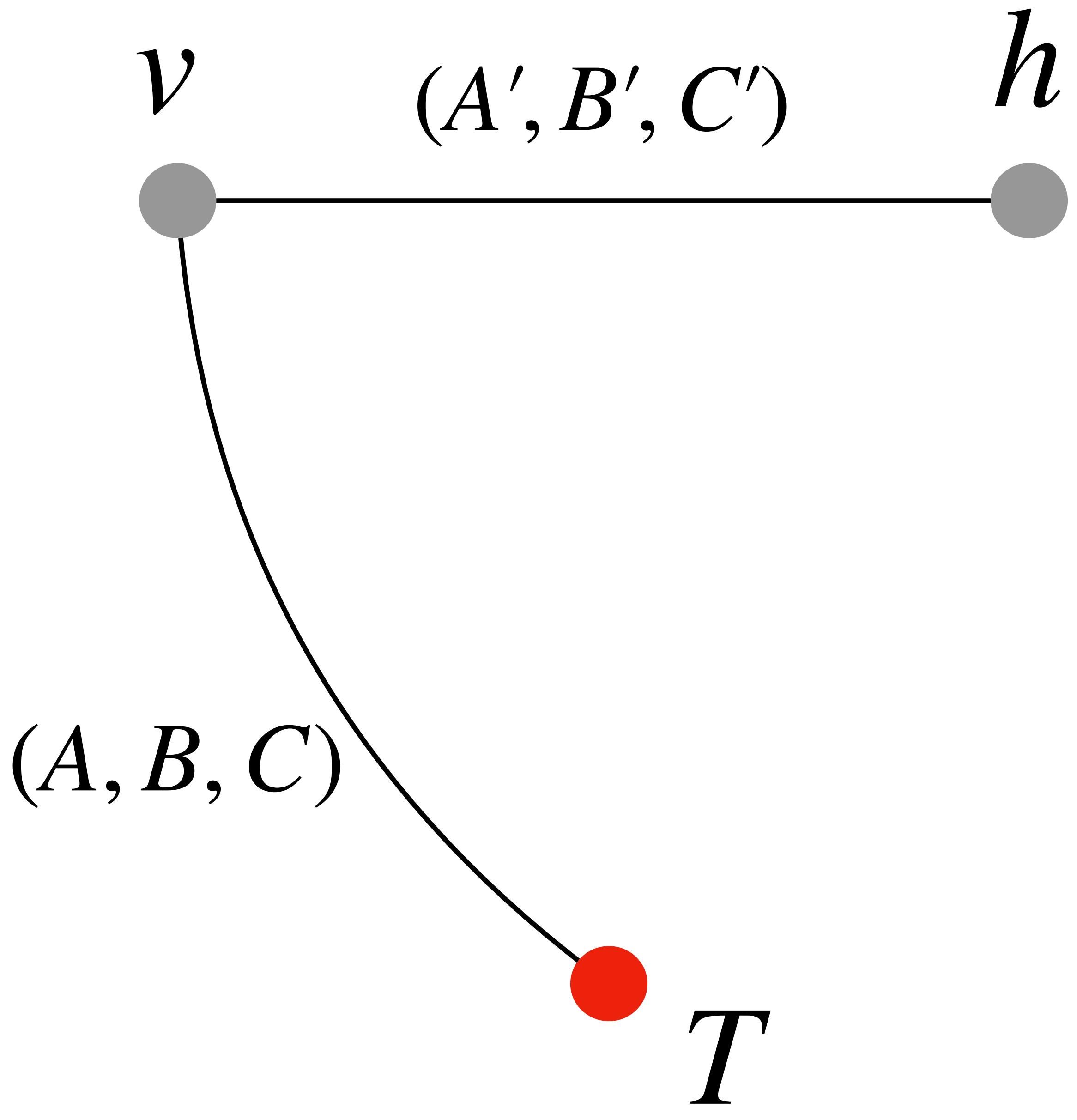

Prover:

v, h, T →

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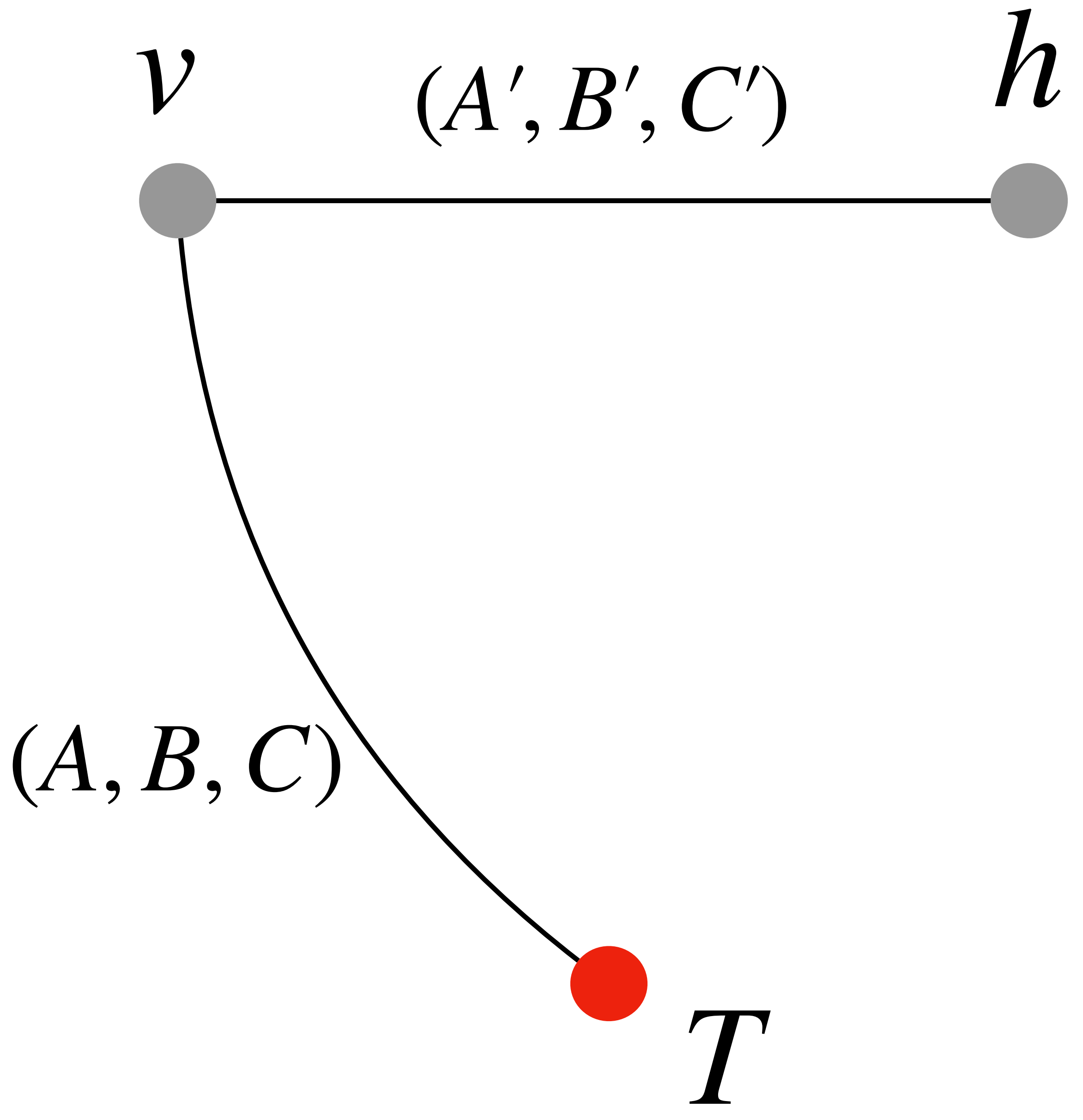

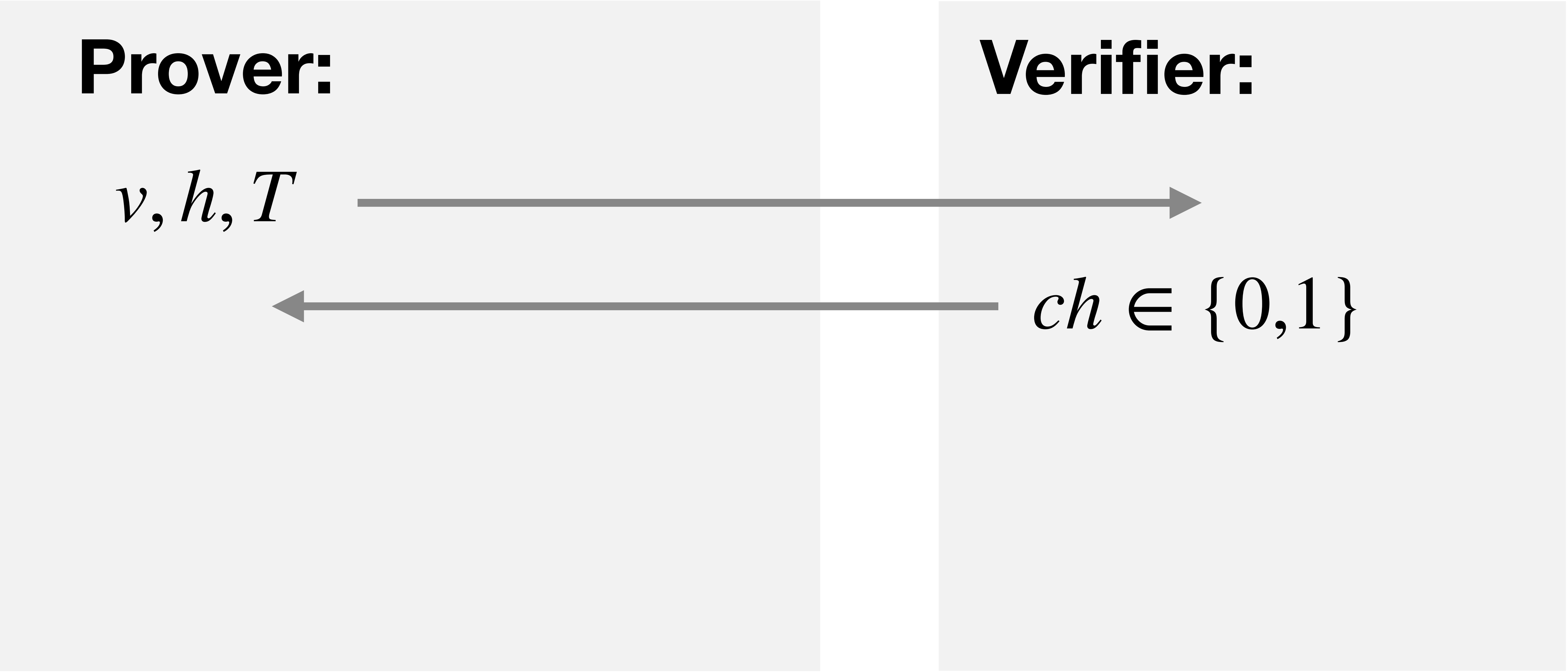

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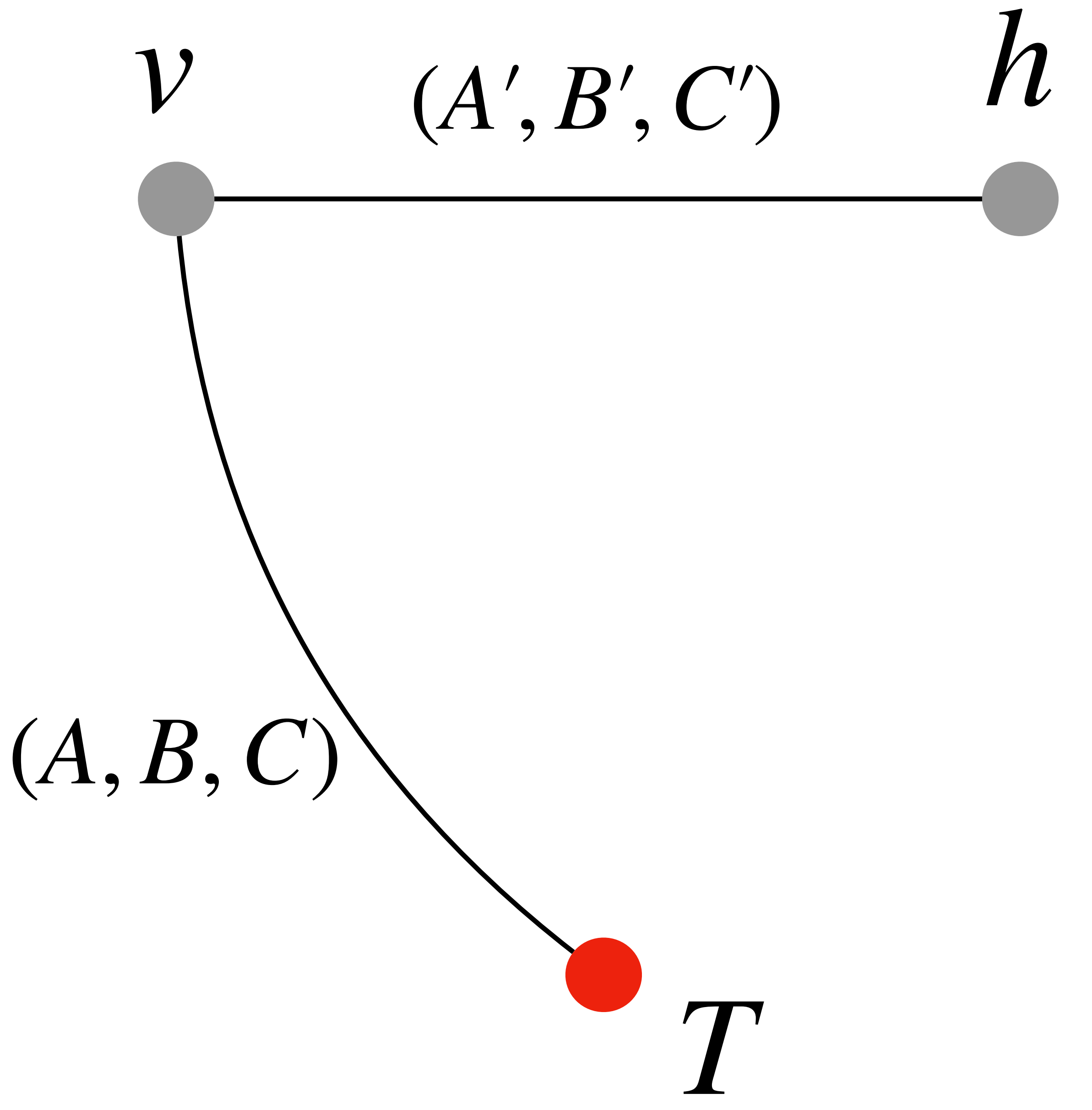

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← $ch \in \{0,1\}$

$$\begin{cases} (A', B', C'), & ch = 0 \\ (A'A, B'B, C'C), & ch = 1 \end{cases}$$ →

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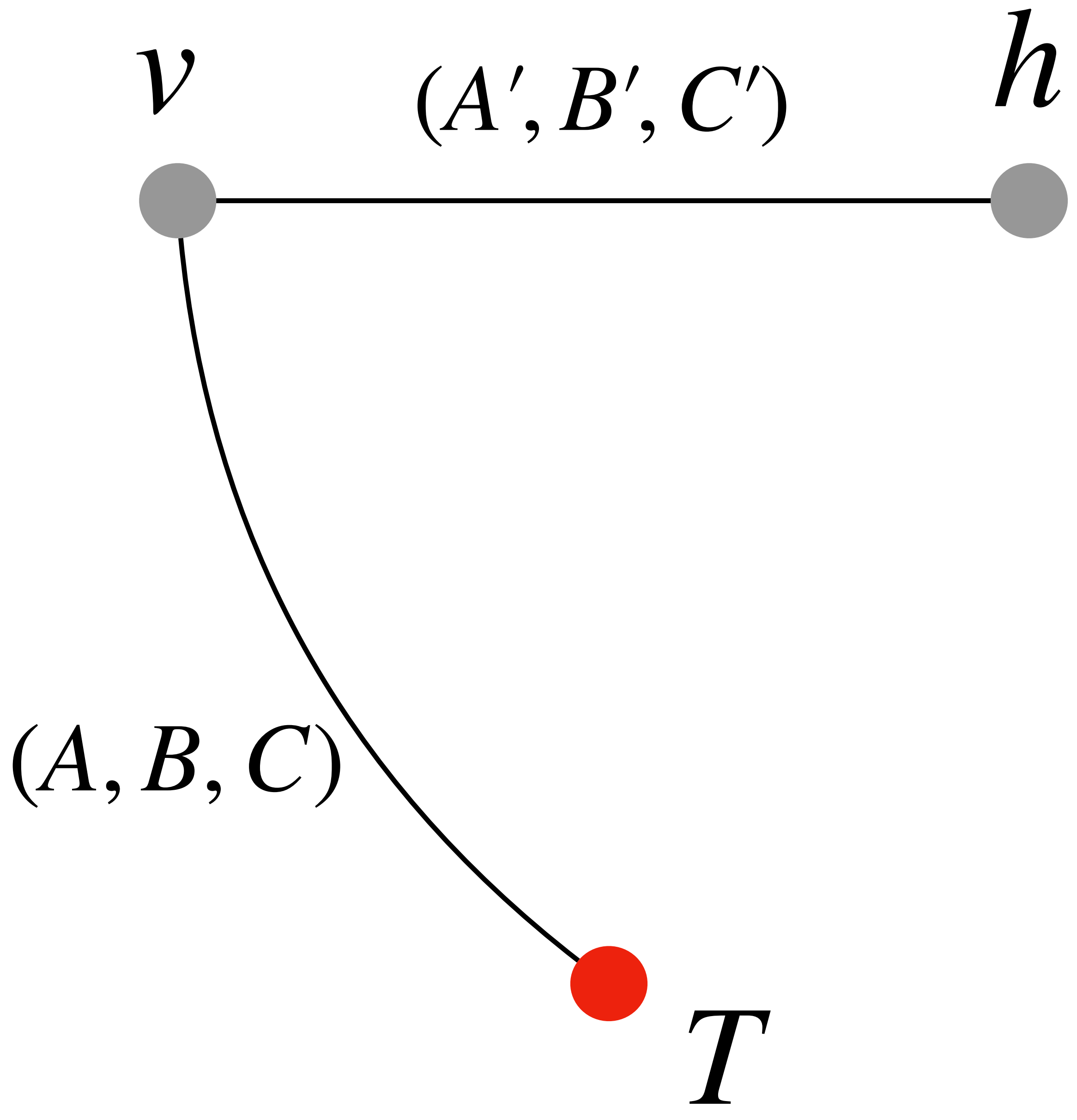

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→ we need to keep v secret

Proofs of knowledge

(A, B, C, b) secret

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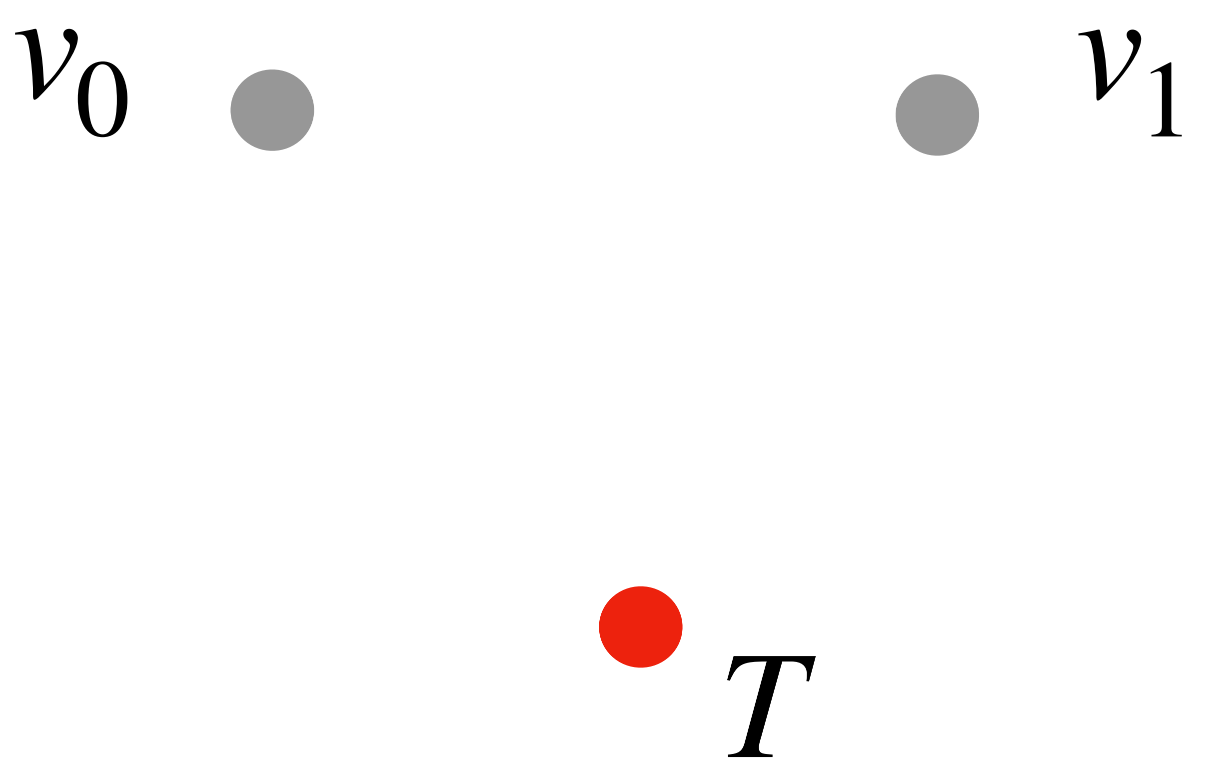

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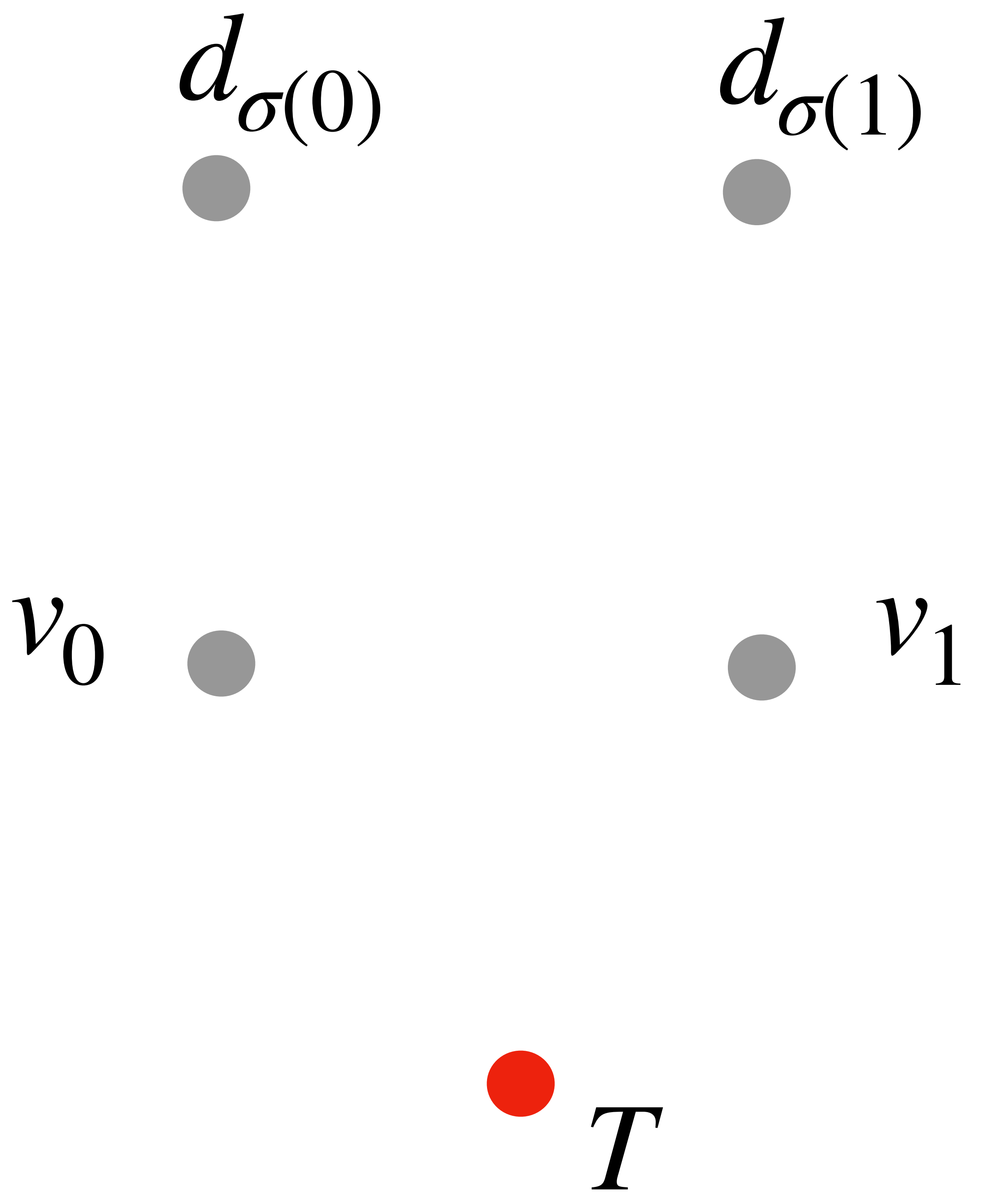

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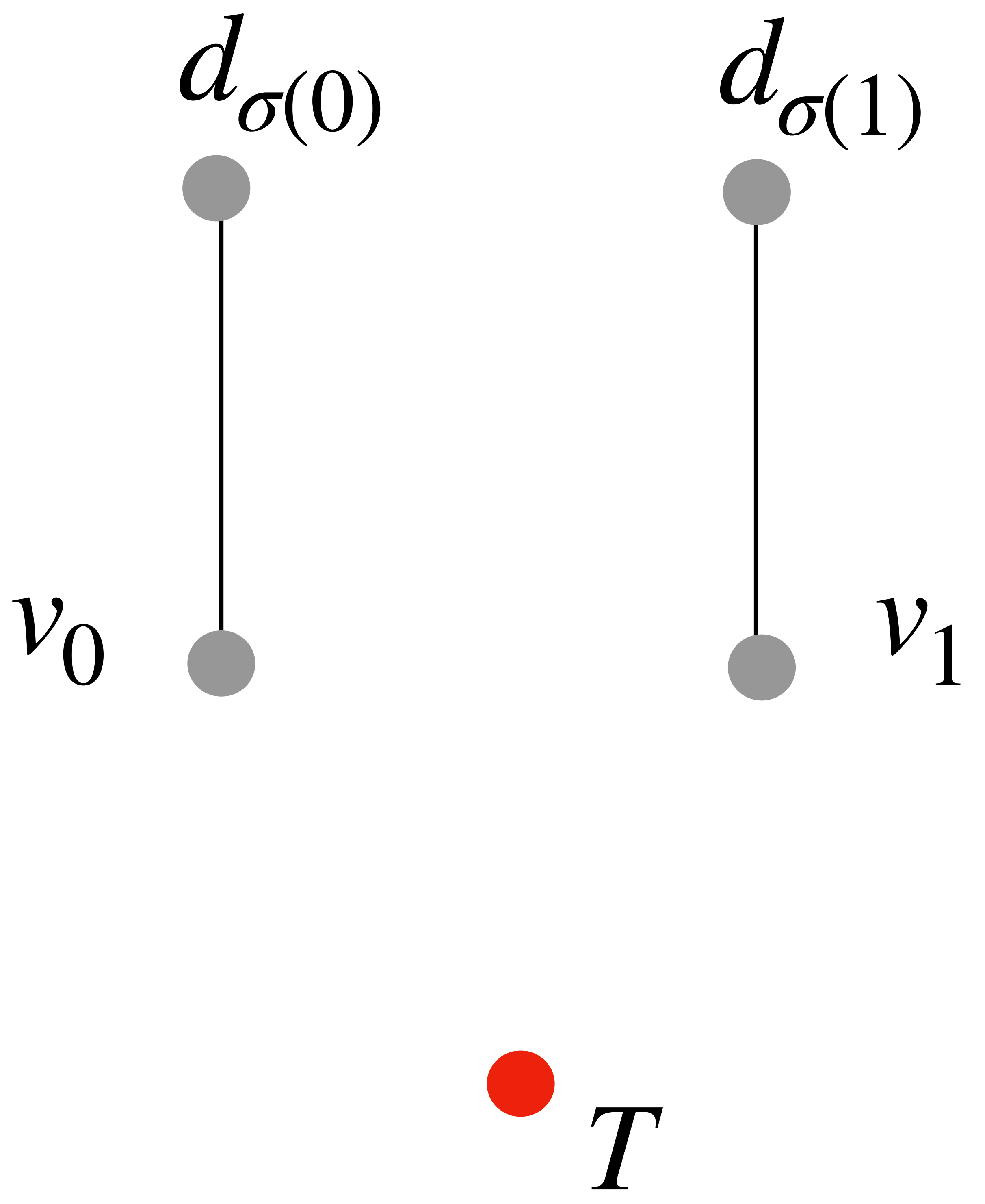

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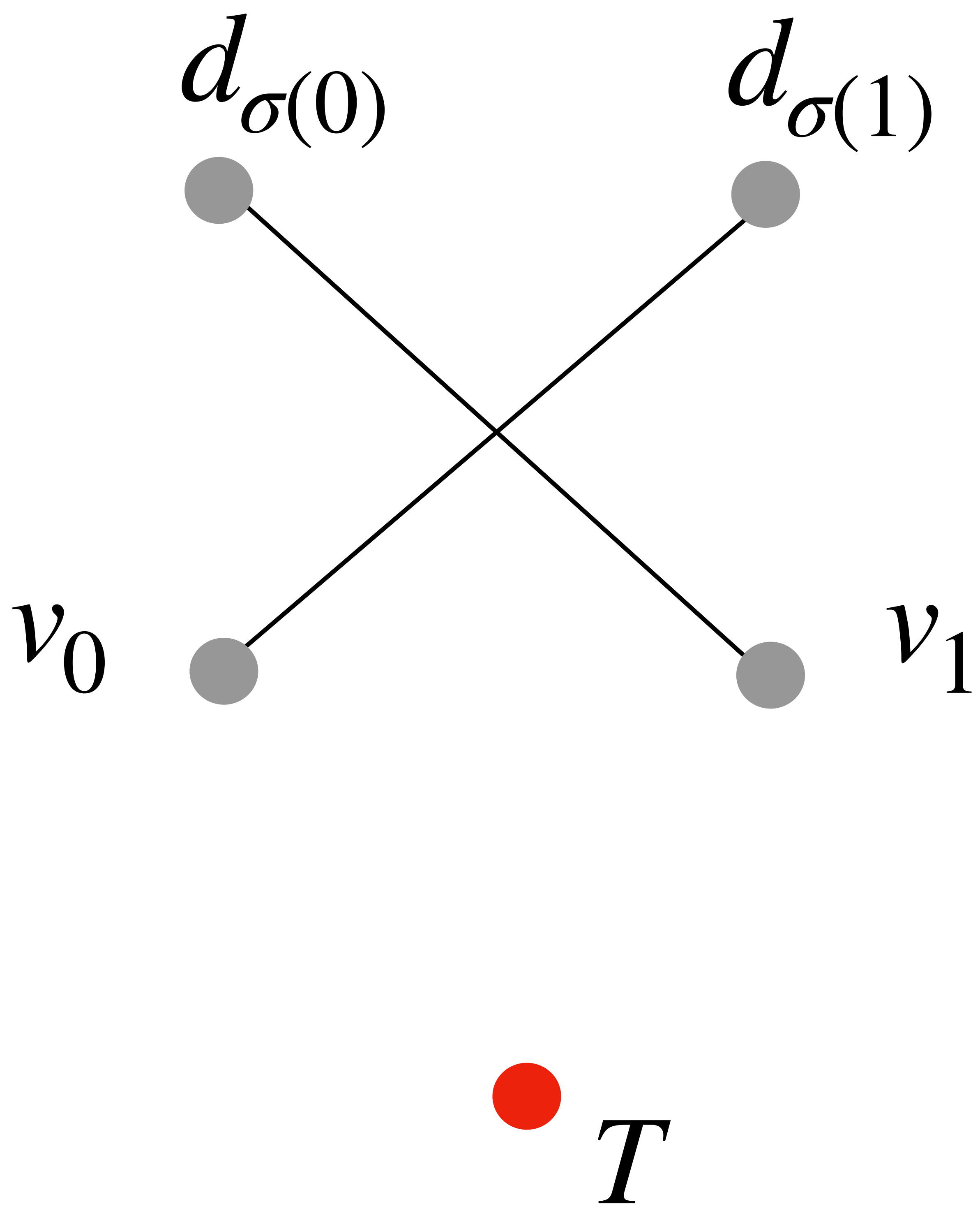

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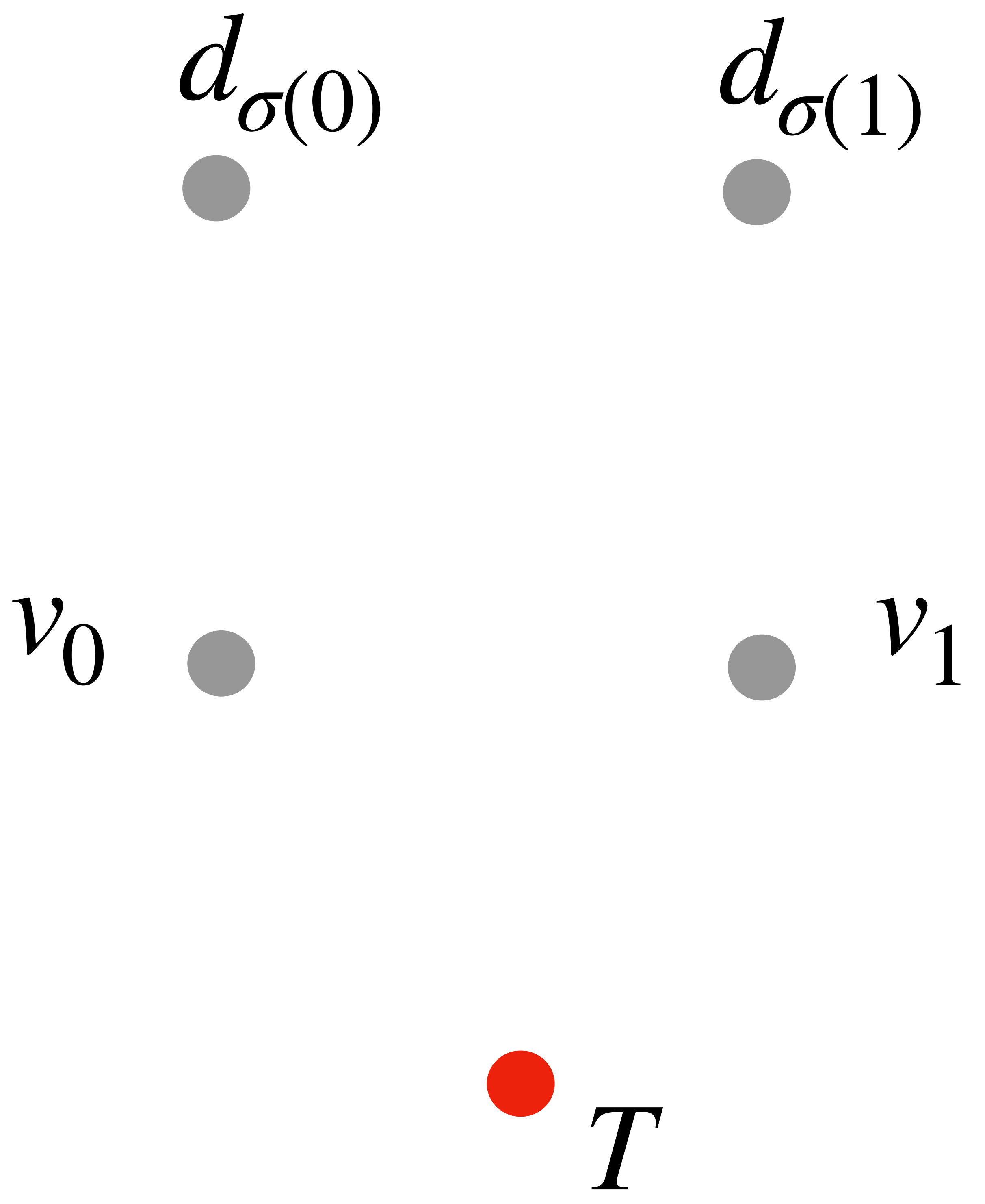

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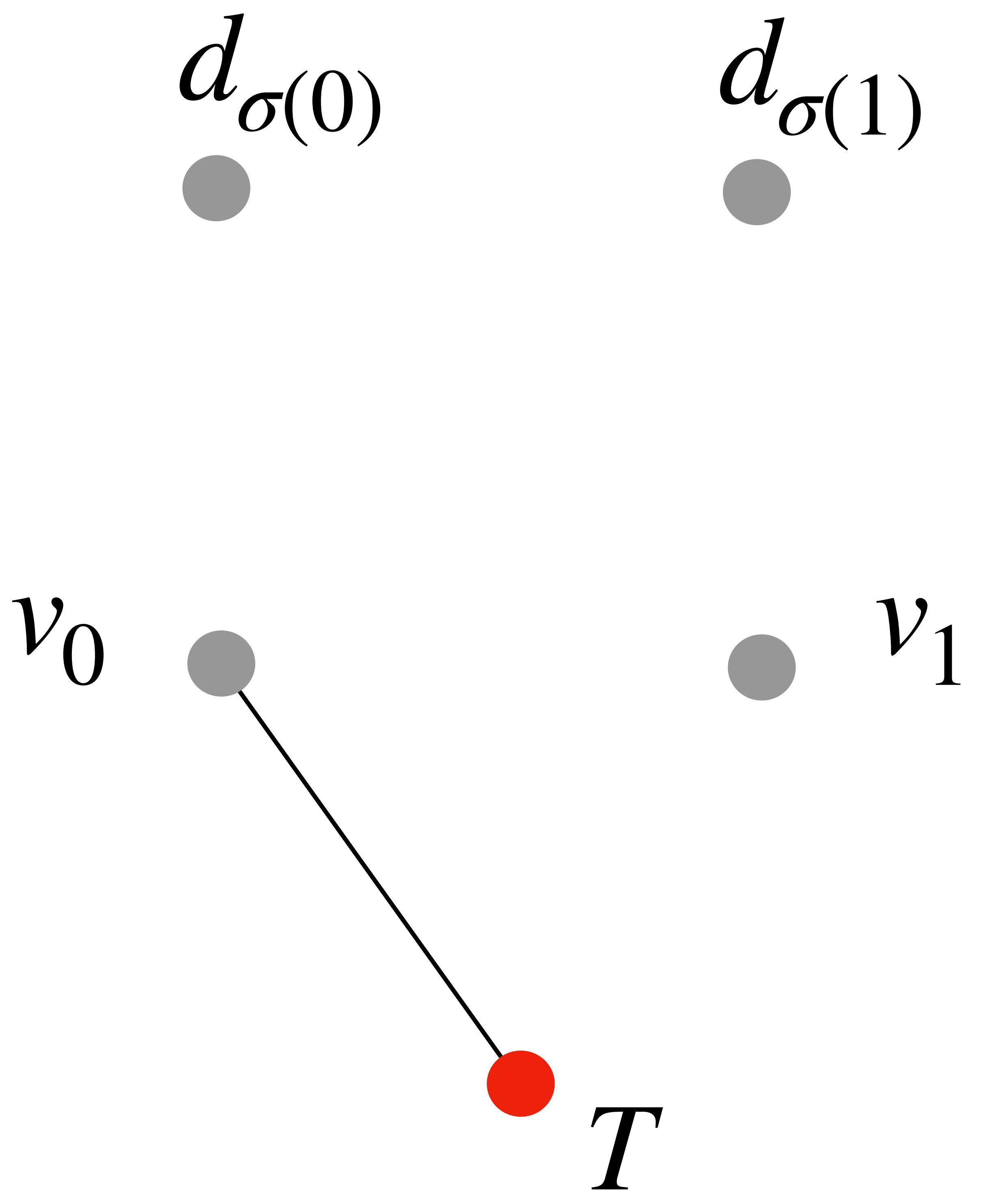

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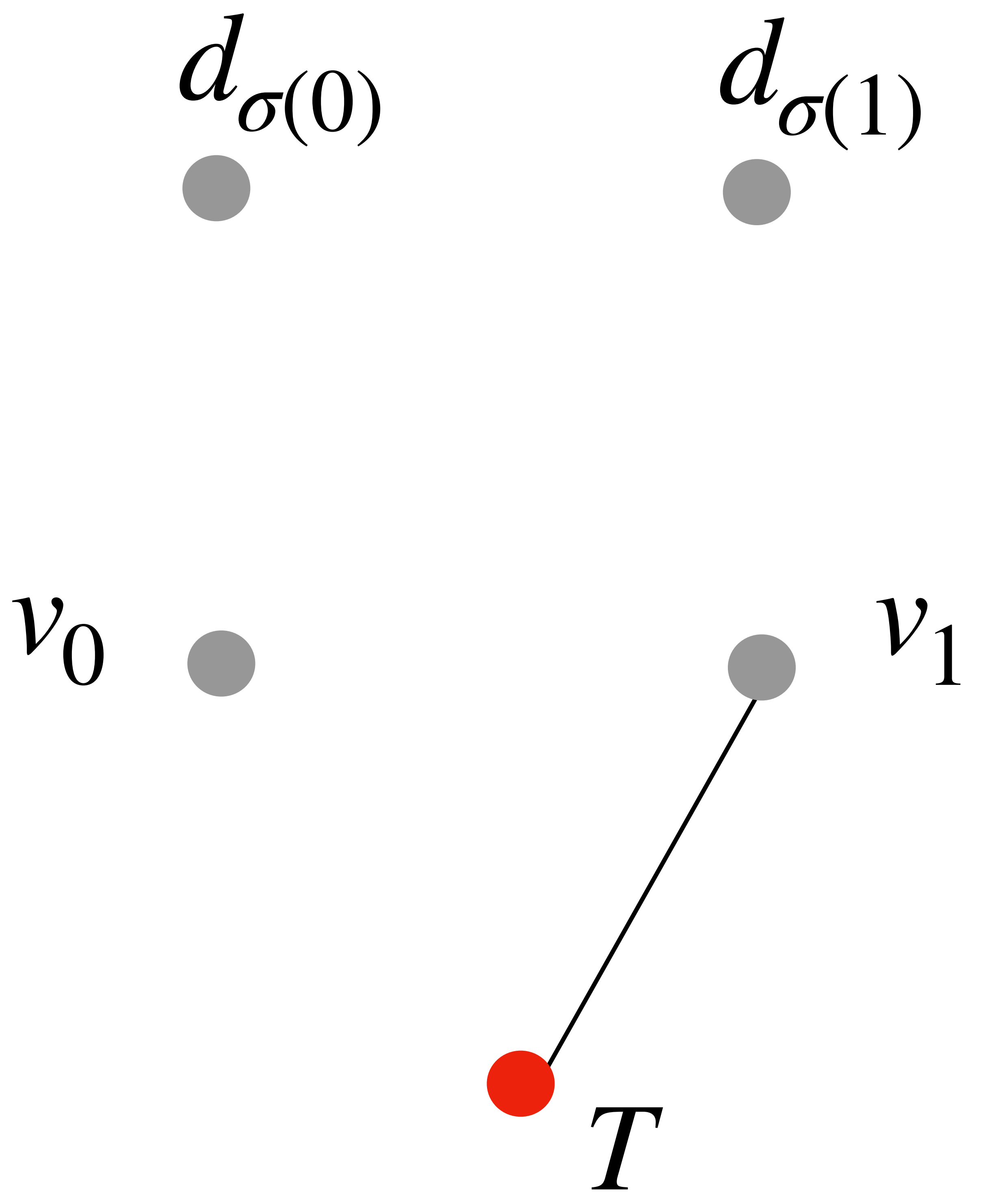

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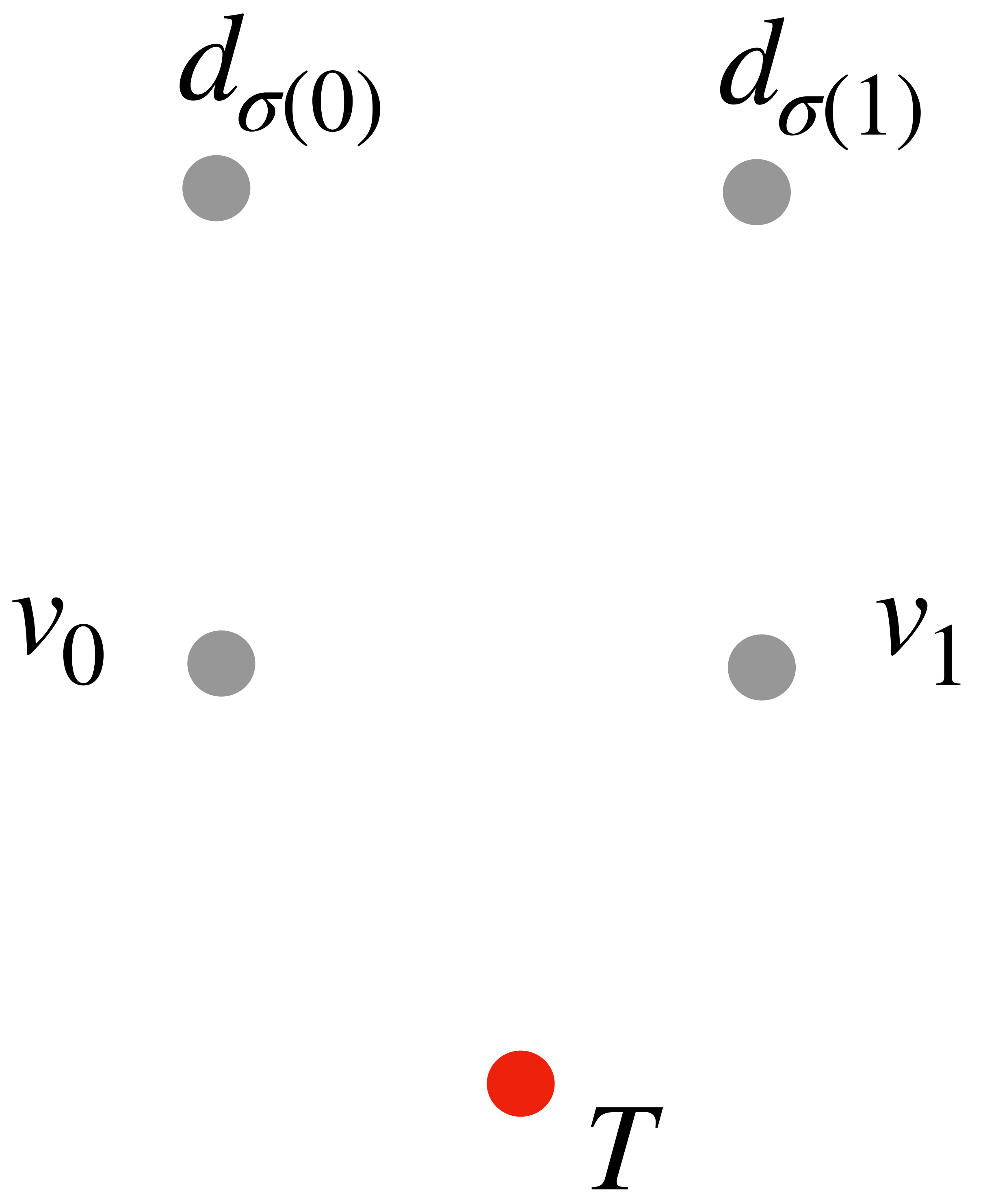

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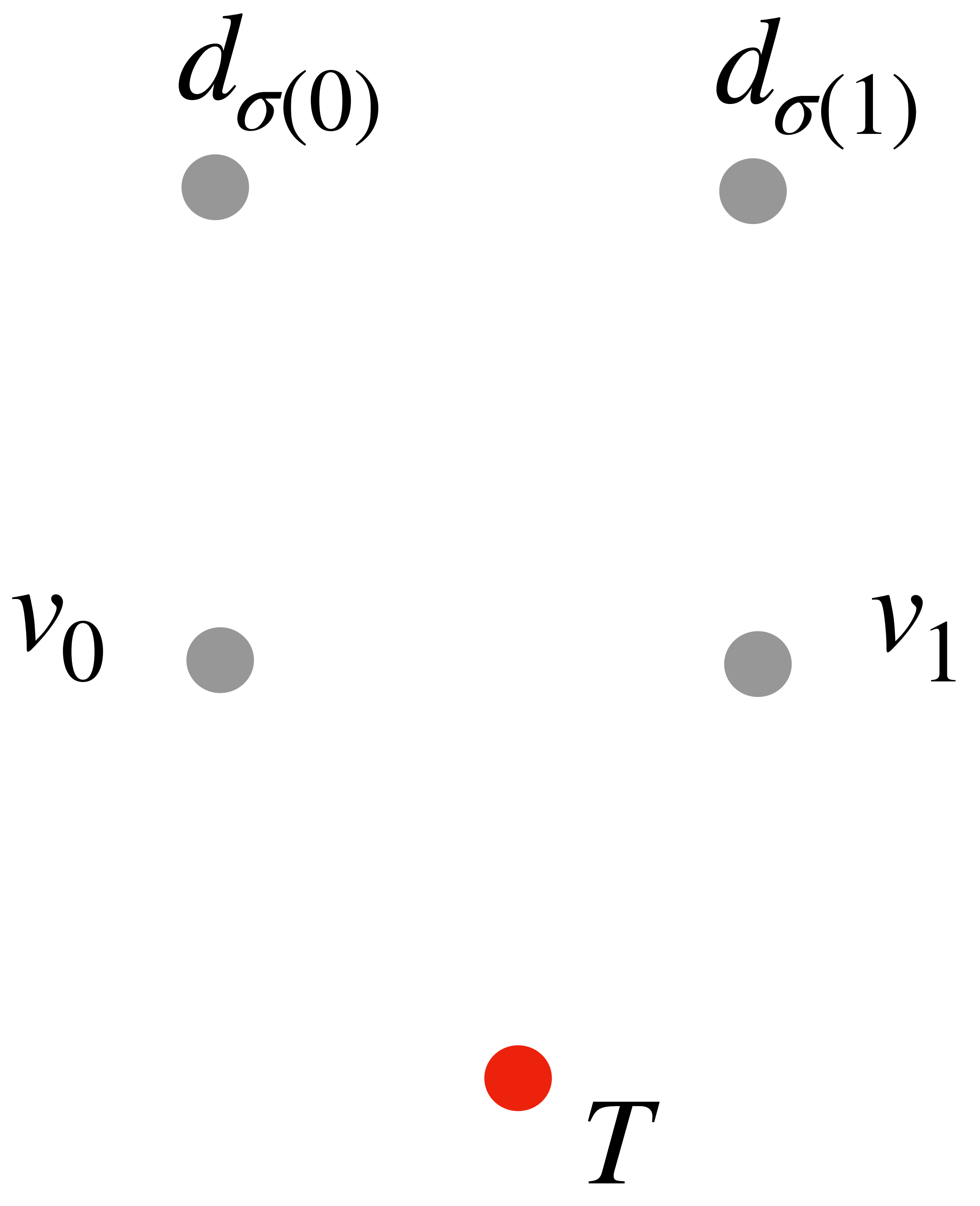

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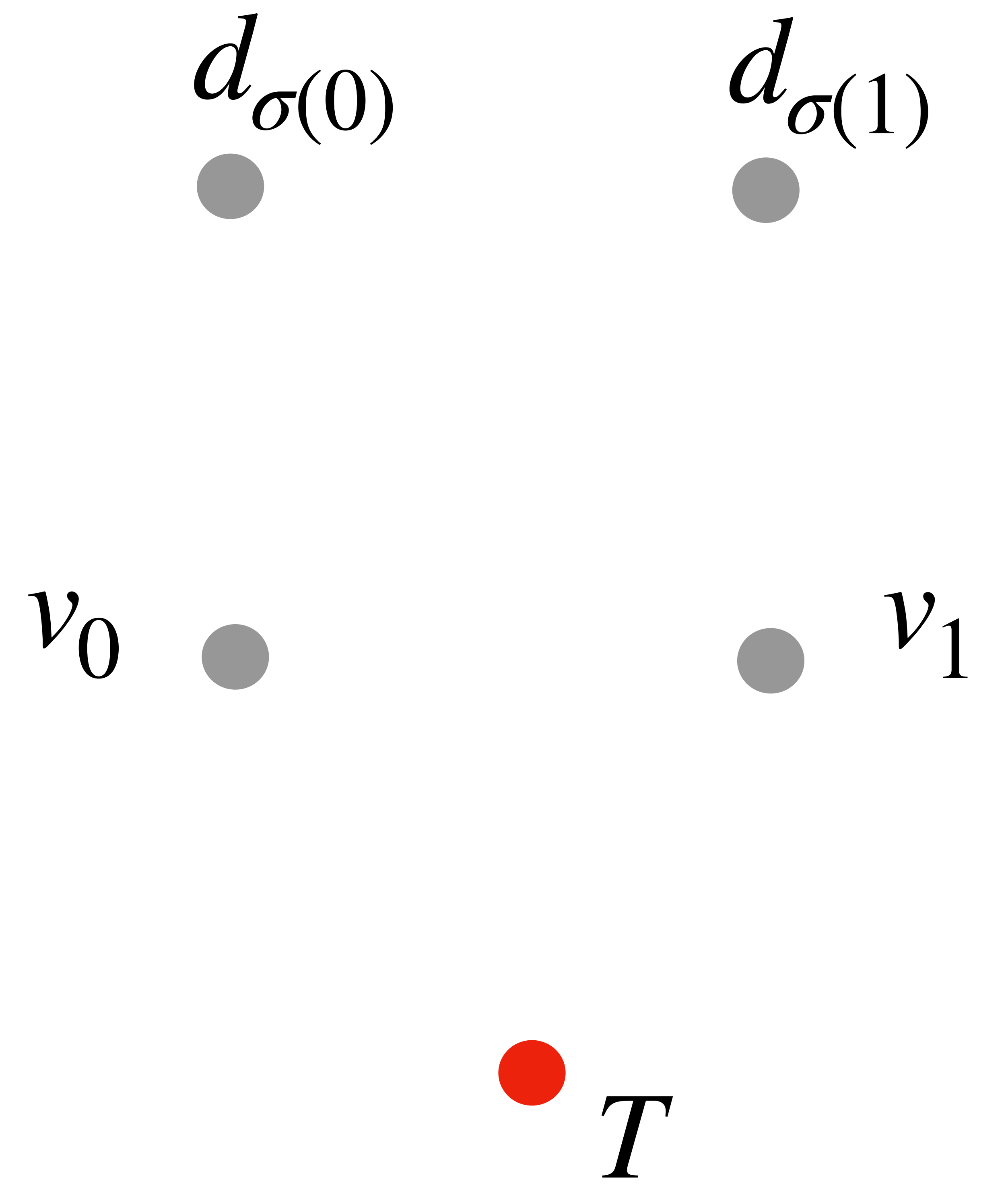
Verifier:

Proofs of knowledge



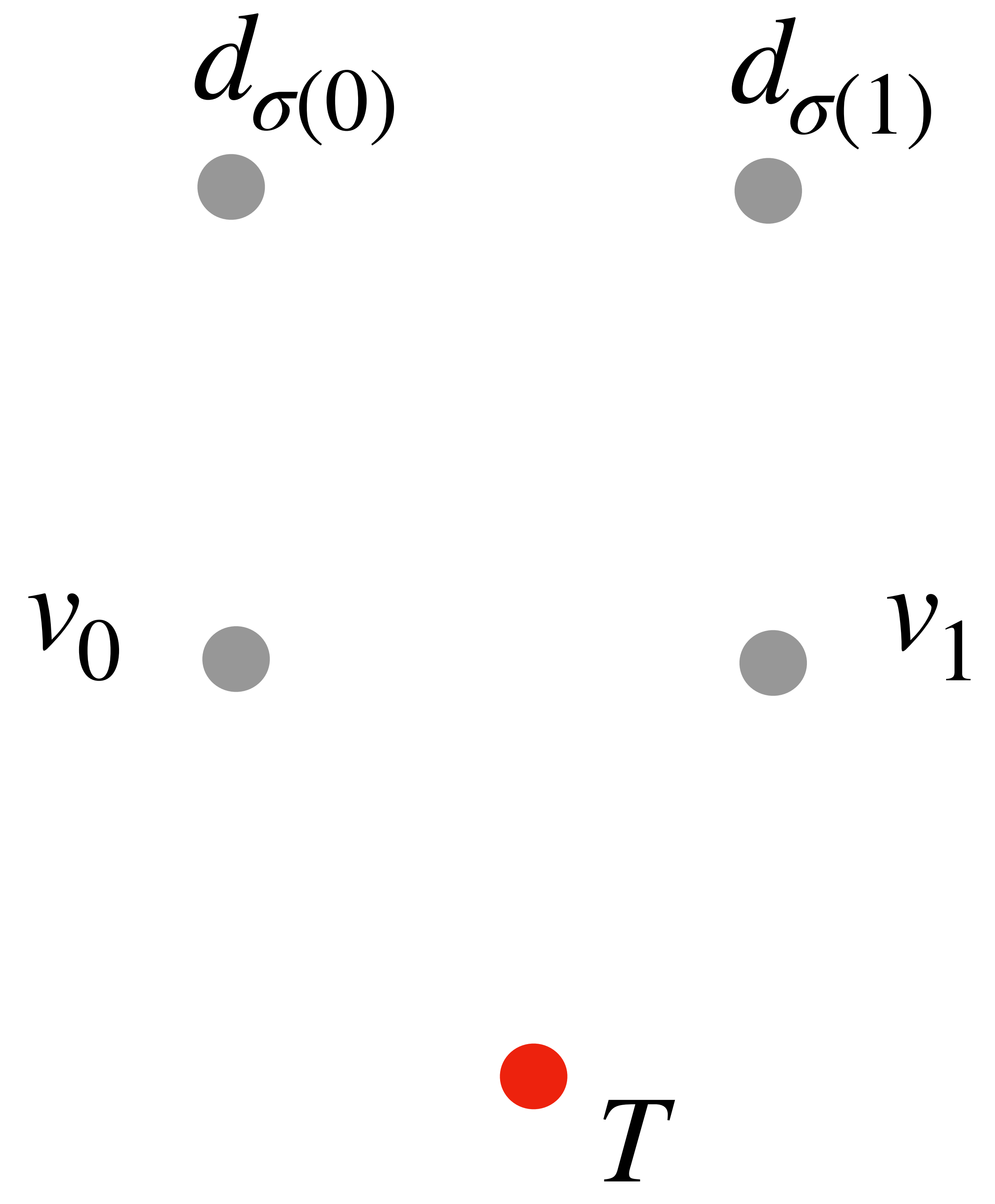
(A, B, C, b) secret

Prover:

$(v_0, v_1), (d_{\sigma(0)}, d_{\sigma(1)}), T$ →

Verifier:

Proofs of knowledge



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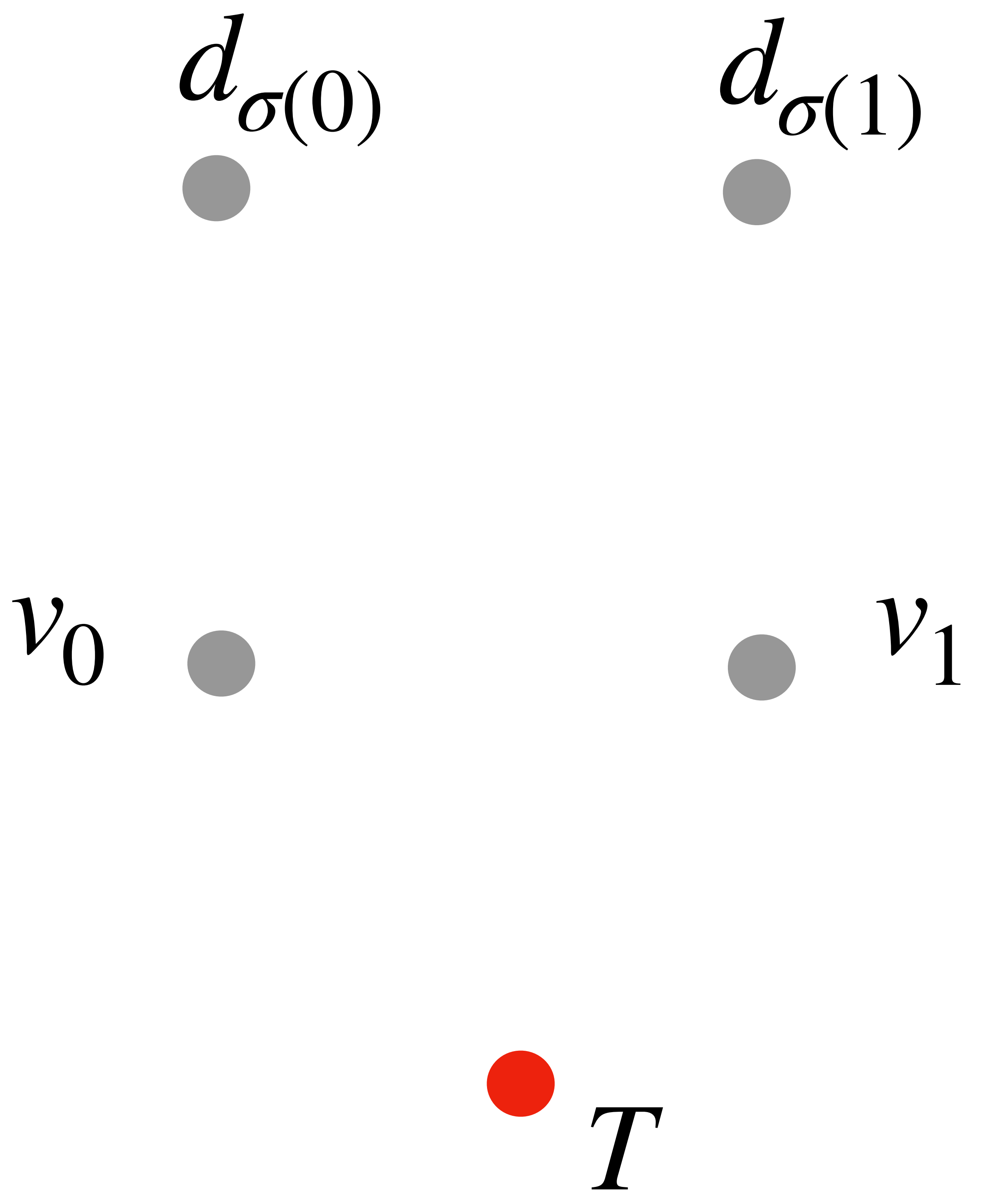

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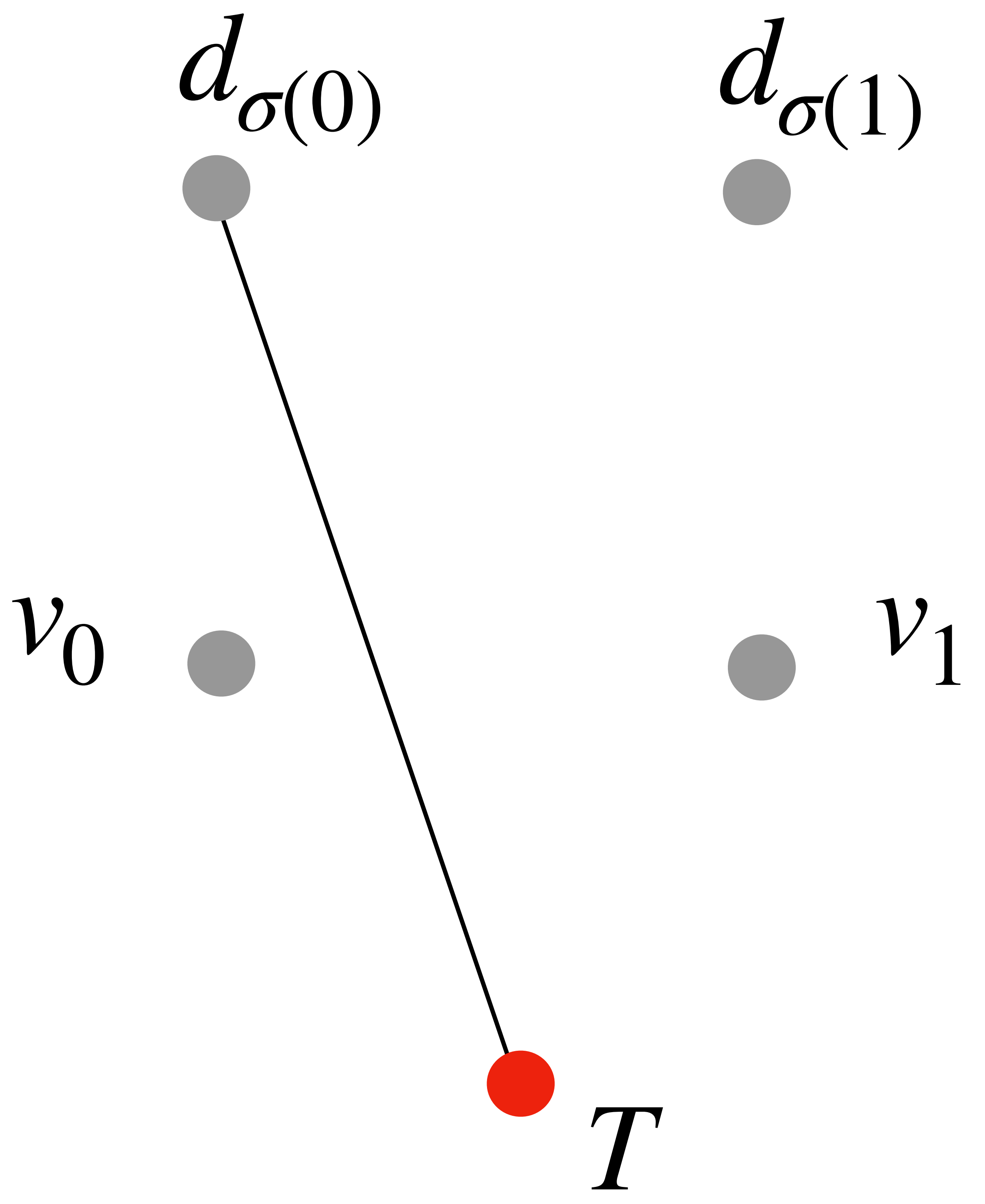

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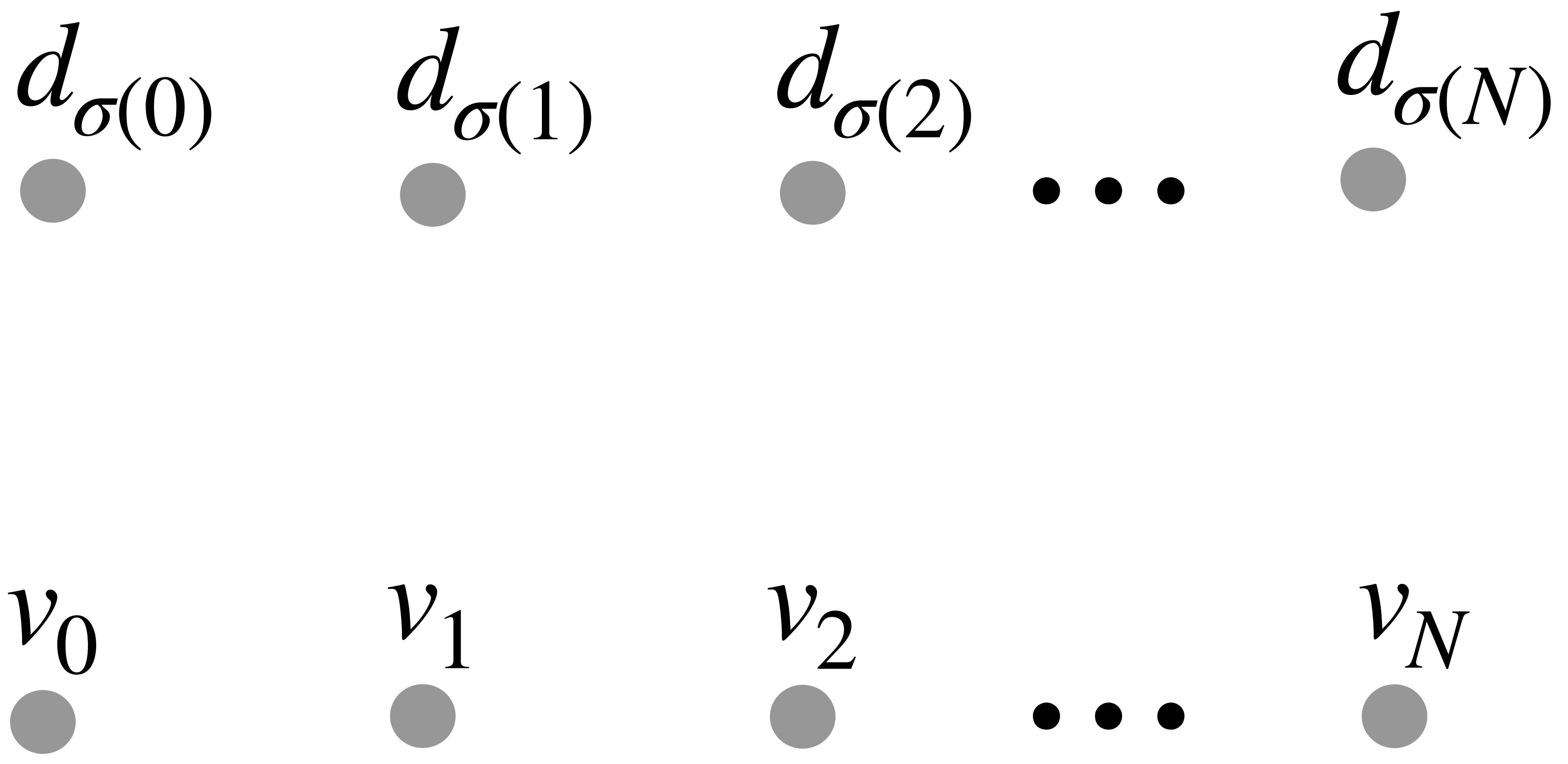
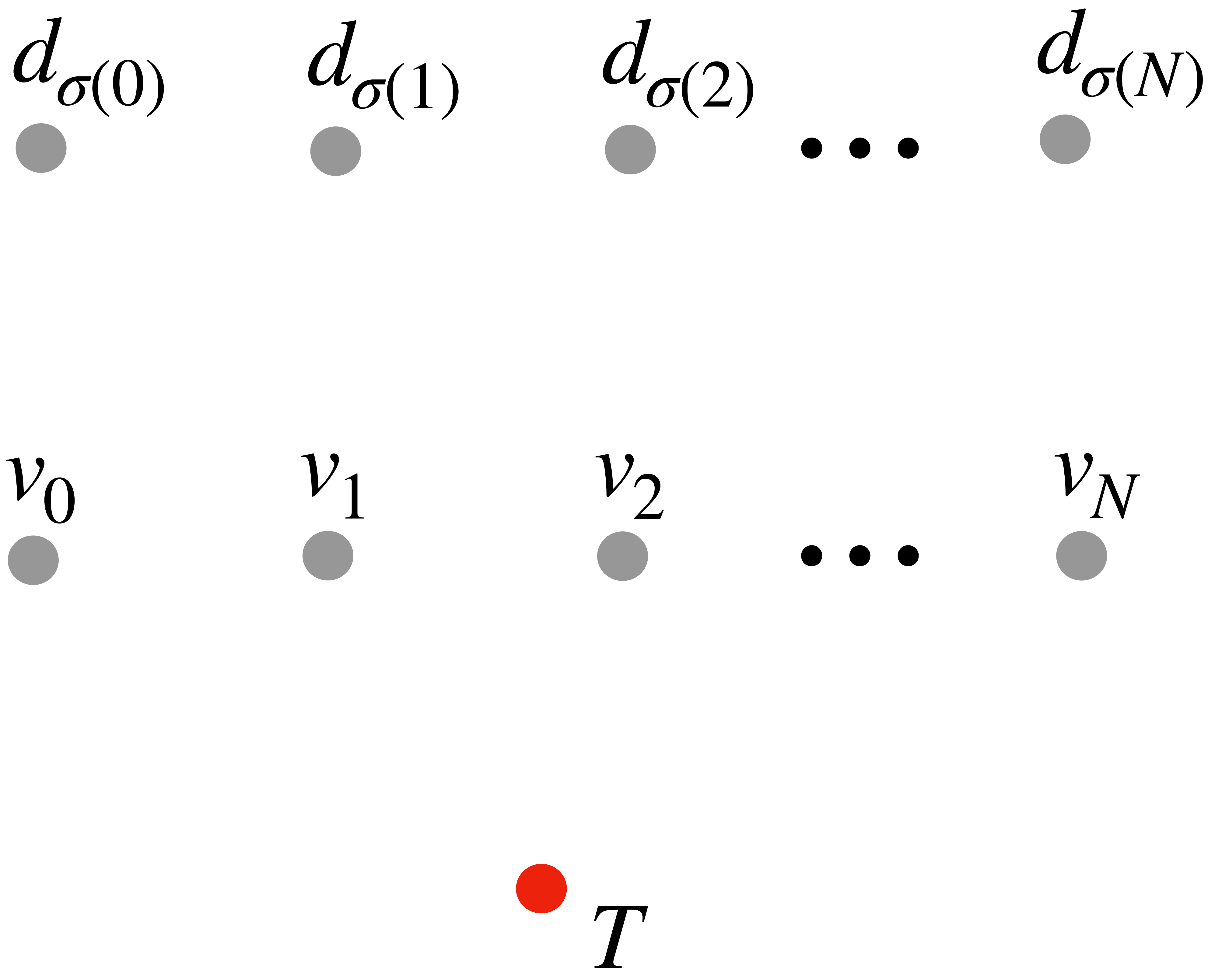


(A, B, C, b) secret

→ if $ch = 0$ then reveal the isomorphisms between v_i and $d_{\sigma(i)}$

→ if $ch = 1$ then reveal the isomorphism from $d_{\sigma(b)}$ to T

Proofs of knowledge



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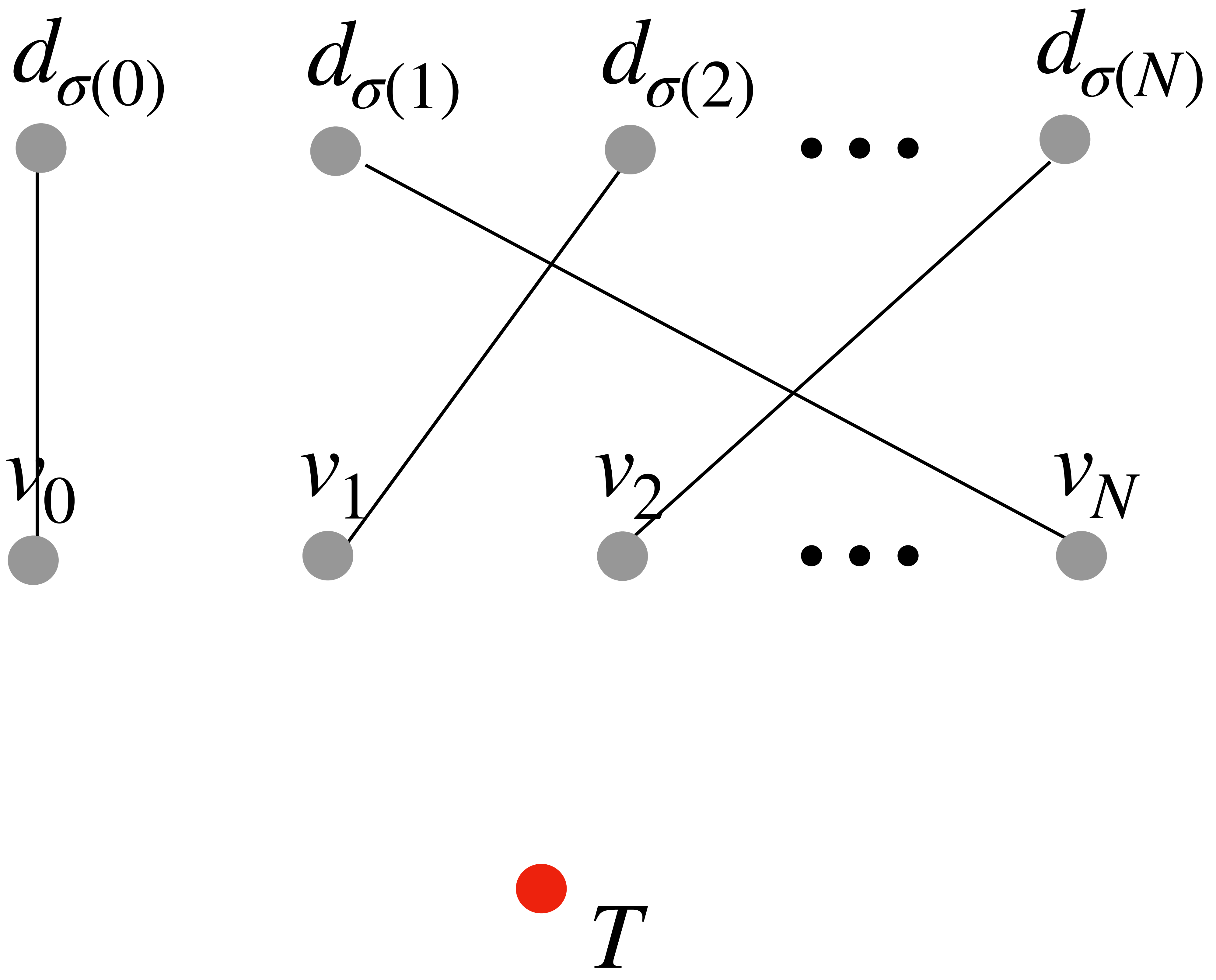

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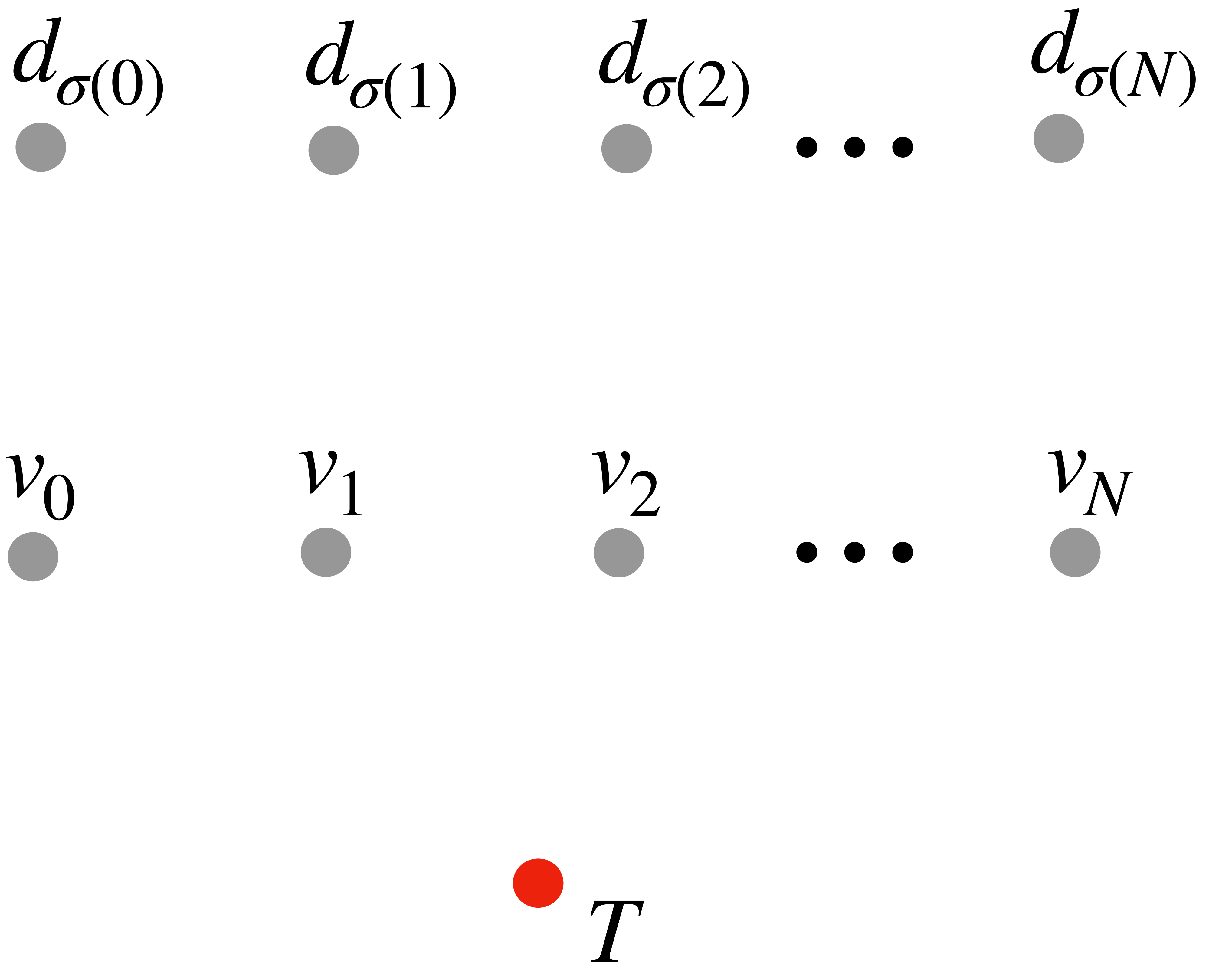

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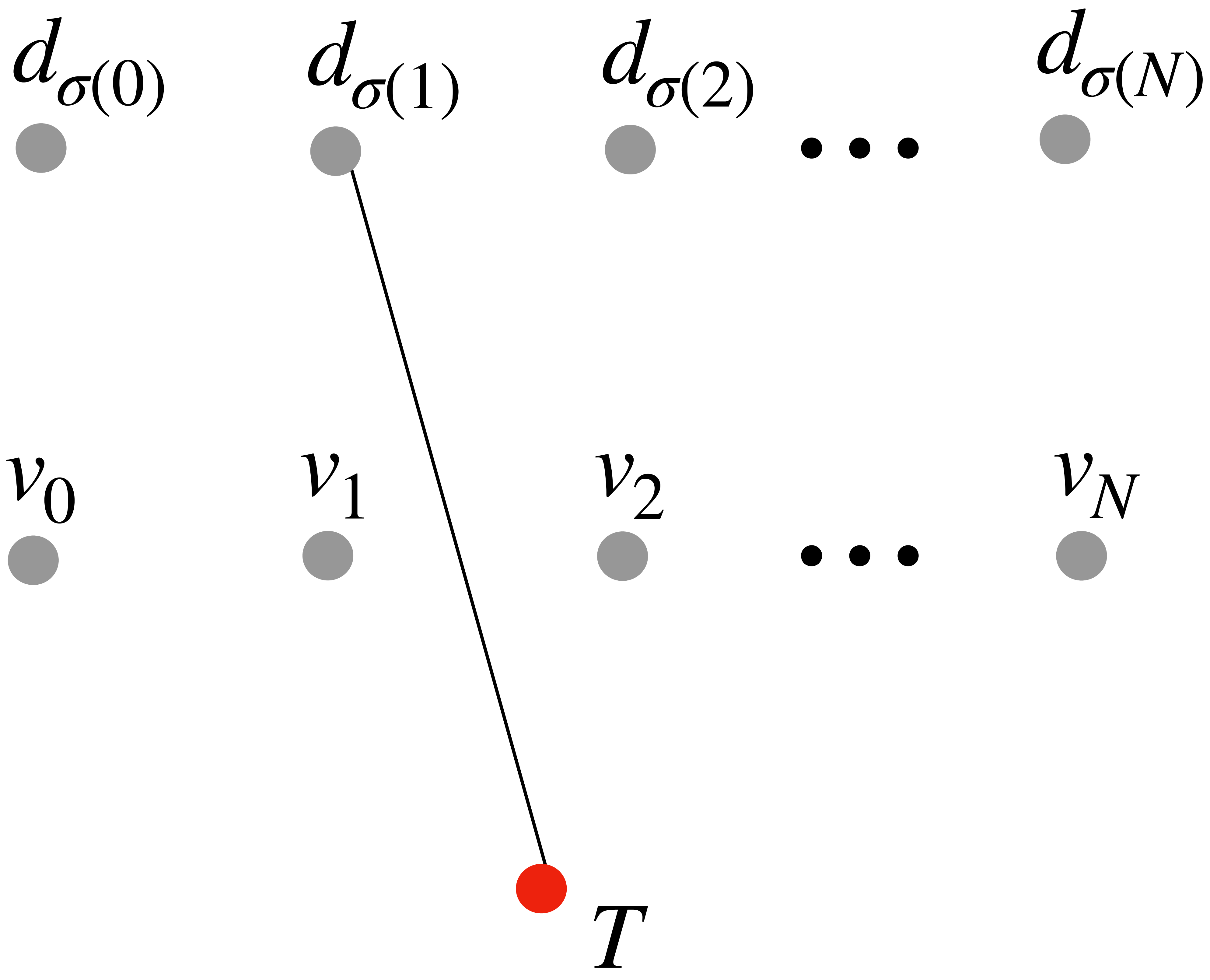

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Solving the Tensor Isomorphism Problem

for special orbits with low rank points:

Cryptanalysis and repair

of an Asiacrypt 2023 commitment scheme

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Thank you!