Computing Rational Isogenies with Irrational Kernel Points

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Separable isogenies—why do we care?

A *separable* isogeny can be defined uniquely (up to isomorphism) by its kernel.

With a cyclic kernel, this means we require only one point (the generator of the kernel) to define an isogeny.

Isogeny-based cryptosystems such as CSIDH and CRS evaluate points at separable isogenies.

Irrational kernels

We want to compute an isogeny of degree ℓ, whose domain is $E(\mathbb{F}_q)$.
Usually, we will look for a point of order ℓ to generate its kernel.

Suppose no point P of order ℓ is defined over $E(\mathbb{F}_q)$, so we go to $E(\mathbb{F}_{q^k})$ to find it.
We say $\langle P \rangle$ is *irrational*.

That is, the kernel of our isogeny will be defined over $\mathbb{F}_q$, but its elements will not be.

Vélu requires the kernel points, so this will be costly to run over an extension field.

Kernel polynomials

The *kernel polynomial* of an isogeny is used in Vélu's formulæ, given by

$$D(X) := \prod_{P \in S} (X - x(P))$$

where $S \subset \langle P \rangle$ is any subset such that

$$S \cap -S = \emptyset \qquad \text{and} \qquad S \cup -S = \langle P \rangle \setminus \{0\}\,.$$

Note, D will not always split over the base field, but its coefficients will always be defined over it.

Evaluating the kernel polynomial

Take $\ell = 13$ and $k = 3$.
We want to choose a set $S \subset \langle P \rangle$, that contains all multiples of P up to negation (excluding the identity) for use in our kernel polynomial.

Some classic examples are taking the first half or odd multiples.

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$$\begin{array}{cccccccccccc}
P & 2P & 3P & 4P & 5P & 6P & 7P & 8P & 9P & 10P & 11P & 12P \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
P & & 3P & & 5P & & 7P & & 9P & & 11P &
\end{array}$$

Evaluating the kernel polynomial

For example, choose

$$S = \{P, [2]P, [3]P, \ldots, [(\ell - 1)/2]P\}$$

This method would use one `xDBL` to get $[2]P$, and $(\ell - 1)/2 - 2 = (\ell - 5)/2$ `xADD`'s.

Evaluating the kernel polynomial

Another approach...

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$$P \xrightarrow{\times 2} 2P \xrightarrow{\times 2} 4P \xrightarrow{\times 2} 8P \xrightarrow{\times 2} 3P \xrightarrow{\times 2} 6P \xrightarrow{\times 2} 12P \xrightarrow{\times 2} 11P \xrightarrow{\times 2} 9P \xrightarrow{\times 2} 5P \xrightarrow{\times 2} 10P \xrightarrow{\times 2} 7P$$

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This uses $(\ell - 3)/2$ `xDBL`'s.

Evaluating the kernel polynomial

We say n is a *primitive root* modulo ℓ if every a coprime to ℓ can be written as $a = n^i$, for some i.

We get the following lemma:

Lemma

If 2 is a primitive root modulo ℓ then

$$S = \{[2^i]P : 0 \leq i < (\ell - 1)/2\}.$$

This Lemma can be extended to the case where 2 is semi-primitive (has order $\frac{\ell-1}{2}$) with added constraints on ℓ.

How (semi)primitive is 2?

How often is 2 (semi)primitive modulo ℓ? For prime $3 \leq \ell < 10000$, roughly 57%.

	Primes $2 < \ell < 600$										
Yes	**3**	**5**	**7**	**11**	**13**	**19**	**23**	**29**	**37**	**47**	**53**
	59	**61**	**67**	**71**	**79**	**83**	**101**	**103**	**107**	**131**	**139**
	149	**163**	**167**	**173**	**179**	**181**	**191**	**197**	**199**	**211**	**227**
	239	**263**	**269**	**271**	**293**	**311**	**317**	**347**	**349**	**359**	**367**
	373	379	383	389	419	421	443	461	463	467	479
	487	491	503	509	523	541	547	557	563	**587**	599
No	**17**	**31**	**41**	**43**	**73**	**89**	**97**	**109**	**113**	**127**	**137**
	151	**157**	**193**	**223**	**229**	**233**	**241**	**251**	**257**	**277**	**281**
	283	**307**	**313**	**331**	**337**	**353**	397	401	409	431	433
	439	449	457	499	521	569	571	577	593		

Table: Primes in bold appear in the CSIDH-512 parameter set.

Evaluating the kernel polynomial

$$\text{Scalar multiplication :} \quad 1\texttt{xDBL} \quad + \tfrac{\ell-5}{2}\texttt{xADD}$$

$$\text{Doubling :} \quad \tfrac{\ell-3}{2}\texttt{xDBL}$$

Model	`xADD`	`xDBL`
Montgomery	$4M + 2S$	$2M + 2S + 1c$
Short Weierstrass (projective)	$11M + 5S$	$1M + 8S + 1c$

Table: Costs of `xADD` and `xDBL`. Here M and S represent multiplication and squaring, respectively, in $\mathbb{F}_{q^k}$, while c represents multiplication by a curve constant in $\mathbb{F}_q$

When $k = 1$, we get $M \approx c$, so we get a saving of around 16%.

Exploiting the action of Frobenius

P is a point of order ℓ defined over $E(\mathbb{F}_{q^k})$ (and not over any proper subfield containing $\mathbb{F}_q$).

$\langle P \rangle$ is Galois stable, so Frobenius acts as an eigenvalue, λ, on $\langle P \rangle$.

$$\begin{array}{ccccccccc} P & \mapsto & \pi(P) & \mapsto & \pi^2(P) & \mapsto & \cdots & \pi^{k-1}(P) \\ P & \mapsto & \lambda P & \mapsto & \lambda^2 P & \mapsto & \cdots & \lambda^{k-1} P \end{array}$$

This λ must be nonzero, and in fact it must be an element of order k in $(\mathbb{Z}/\ell\mathbb{Z})^\times$.

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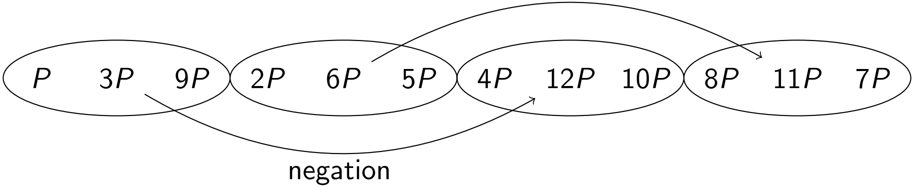

Exploiting the action of Frobenius

So we can write

$$S = S_0 \sqcup \pi(S_0) \sqcup \cdots \sqcup \pi^{k'-1}(S_0)$$

where S_0 is the set of points that generate the galois orbits (up to negation) and

$$k' := \begin{cases} k & \text{if } k \text{ is odd}, \\ k/2 & \text{if } k \text{ is even}. \end{cases}$$

Lemma

If 2 is primitive modulo ℓ (or some other conditions) then $S_0 = \{[2^i]P : 0 \leq i < (\ell - 1)/2k'\}$.

From before, let $\ell = 13, k = 3$.
This gives us $S_0 = \{[2^i]P : 0 \leq i < 12/6\} = \{P, 2P\}$.

Evaluating the kernel polynomial with S_0

$$D(X) = \prod_{P\in S}(X - x(P)) = \prod_{P\in S_0}\prod_{i=0}^{k'-1}(X - x(\pi^i(P)))$$

$$= \prod_{P\in S_0}\prod_{i=0}^{k'-1}(X - x(P)^{q^i}),$$

Transposing the order of the products, if we let

$$D_0(X) := \prod_{P\in S_0}(X - x(P))$$

then

$$D(\alpha) = (D_0(\alpha))^{1+2+\cdots(k'-1)} \quad \text{for all } \alpha \in \mathbb{F}_q .$$

Some preliminary results

We use an algorithm due to Costello and Hisil (2017) as a base for comparison.

They derive the following affine formula

$$f(x) = x \cdot \left(\prod_{i=1}^{(\ell-1)/2} \left(\frac{x \cdot x_{[i]P} - 1}{x - x_{[i]P}} \right) \right)^2$$

which results in this projective formula

$$X' = X \cdot \left(\prod_{i=1}^{(\ell-1)/2} (X \cdot X_i - Z_i \cdot Z) \right)^2, Z' = Z \cdot \left(\prod_{i=1}^{(\ell-1)/2} (X \cdot Z_i - X_i \cdot Z) \right)^2$$

This work was later generalized to the even case by Renes (2018).

Some preliminary results

We compare the algorithms using operation counts over the extension field.

Some examples **when 2 is primitive**...

ℓ	k		multiplies	adds	squarings
7	3	Costello-Hisil	20	17	7
		Galois orbits	2	4	8
29	7	Costello-Hisil	108	105	29
		Galois orbits	8	12	48
53	13	Costello-Hisil	204	201	53
		Galois orbits	8	18	107

Computing the kernel generator

We want to find a point $P \in E(\mathbb{F}_{q^k})$ of order ℓ.

One approach would be:

1. sample a random $P \in E(\mathbb{F}_{q^k})$
2. compute $P_\ell = [\#E(\mathbb{F}_{q^k})/\ell]P$
3. compute $\mathrm{Order}(P_\ell)$, which is either 0 or ℓ

Computing the kernel generator

Take $k = 3$.

$$E(\mathbb{F}_{q^3}) \cong E(\mathbb{F}_q) \oplus H_3$$

Note, π^3 fixes all points in $E(\mathbb{F}_{q^3})$, and π fixes all the points in $E(\mathbb{F}_q)$.
That is, $E(\mathbb{F}_{q^i}) = \ker(\pi^i - [1])$.

So we proceed by:

1. sample $P \in E(\mathbb{F}_{q^k})$
2. compute $P_H = (\pi - 1)P$
3. compute $P_\ell = [\#H_3/\ell]P_H$
4. check the order of P_ℓ

This saves us around $1/3$ of the multiplications.

Computing the kernel generator

Take $k = 6$.

$$\begin{tikzcd}[row sep=small] & E(\mathbb{F}_{q^2}) \arrow[dr, phantom, "\subset" sloped] & \\ E(\mathbb{F}_q) \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & E(\mathbb{F}_{q^6}) \\ & E(\mathbb{F}_{q^3}) \arrow[ur, phantom, "\subset" sloped] & \end{tikzcd}$$

$$E(\mathbb{F}_{q^2}) = \ker(\pi^2 - [1])$$
$$E(\mathbb{F}_{q^3}) = \ker(\pi^3 - [1])$$

$$\delta_6 := (\pi + [1])(\pi^3 - [1])$$

As before, we do the following:

1. sample $P \in E(\mathbb{F}_{q^k})$
2. compute $P_H = \delta_6 P$
3. compute $P_\ell = [\#H_6/\ell]P_H$
4. check the order of P_ℓ

This saves around $2/3$ of the multiplications.

Computing the kernel generator

Formalizing...

$$\eta_k := \Phi_k(\pi) \in \mathrm{End}(E)$$

where $\Phi_k(X)$ is the k-th cyclotomic polynomial (that is, the minimal polynomial over $\mathbb{Z}$ of the primitive k-th roots of unity in $\bar{\mathbb{Q}}$).

So we define

$$\delta_k := (\pi^k - [1])/\eta_k \in \mathrm{End}(E)$$

and get that $\delta_k(E(\mathbb{F}_{q^k})) \subset H_k$.

Computing the kernel generator

Lemma

If k is even, then every point P in H_k has $x(P)$ in $\mathbb{F}_{q^{k/2}}$.

In the case $k = 2$, this is known as the "twist trick".

Computing the kernel generator

k	$\#H_k$	δ_k
1	1	
2	$q + O(\sqrt{q})$	$\pi - [1]$
3	$q^2 + O(q^{3/2})$	$\pi - [1]$
4	$q^2 + O(q)$	$\pi^2 - [1]$
5	$q^4 + O(q^{7/2})$	$\pi - [1]$
6	$q^2 + O(q^{3/2})$	$(\pi + [1])(\pi^3 - [1])$
7	$q^6 + O(q^{7/2})$	$\pi - [1]$
8	$q^4 + O(q^2)$	$\pi^4 - [1]$
9	$q^6 + O(q^{4/2})$	$\pi^3 - [1]$
10	$q^4 + O(q^{7/2})$	$(\pi + [1])(\pi^5 - [1])$
11	$q^{10} + O(q^{19/2})$	$\pi - [1]$
12	$q^4 + O(q^3)$	$(\pi^2 + [1])(\pi^6 - [1])$

In summary we...

Evaluated kernel polynomials

- doubling trick (and its cost)
- exploiting the action of frobenius

Computed the kernel generator

eprint coming soon...